ON THE SEMI-CLASSICAL LIMIT FOR THE LANDAU-FERMI-DIRAC EQUATION

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ABSTRACT. We study sequences of solutions to the inhomogeneous Landau-Fermi-Dirac equation with Coulomb potential in which the quantum parameter converges to zero. Our main result establishes the compactness of these sequences, which allows us to show that, up to a subsequence, these solutions converge to a renormalized solution of the classical Landau equation with a defect measure, as defined by Villani.

To do this, we work in the class of solutions that are obtained through approximation procedures. For these solutions, we were able to show compactness in the vanishing quantum parameter limit through a diagonal argument, which combines techniques from the study of Cauchy problems for both the classical Landau and the Landau-Fermi-Dirac equations.

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1. INTRODUCTION

In 1936 Landau [1] proposed, based on phenomenological arguments, an equation that models the behavior of dilute plasmas, today known as the Landau equation,

$$(1) \qquad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = \mathrm{div}_v \left(\int a(v - v_*)\big(f_* \nabla_v f - f \nabla_{v_*} f_*\big) \, dv_* \right).$$

The unknown $f = f(t, x, v)$ represents the probability density of finding a particle at time t, in position x, with velocity v. Above, we have used the abbreviation f_* for the function $f(t, x, v_*)$ as well as the convention $(\mathrm{div}_v M)_i = \sum_{j=1}^{N} \frac{\partial M_{ij}}{\partial v_j}$ for the divergence of a matrix function M.

The left-hand side is the transport part of the equation, which models the inertial aspect of the particles' movement. In fact, if the right-hand side of the equation were zero, then (1) would reduce

to the free transport equation, and the particles would then follow straight trajectories along the characteristic lines.

The right-hand side, on the other hand, models how these trajectories are affected due to binary collisions between the particles. The matrix $a(v - v_*)$ inside the integral is called the *collision kernel* and typically has the form

$$(2) \qquad a(z) = \Gamma(|z|) \left(I_n - \frac{z \otimes z}{|z|^2} \right),$$

where I_n denotes the $n \times n$ identity matrix and $\Gamma(|z|)$ is an $L_{loc}^p(\mathbb{R}^N)$ function, called the *cross section*, which varies according to the interaction potential between the particles. This way, if the particles interact through a potential of the type

$$U(x) = \frac{k}{|x|^s},$$

then it can be shown that $\Gamma(|z|) = K|z|^\theta$, where $\theta = 3 - 4/s$, for $s \geq 2$ (see, for example, [5]). The cases typically considered in the literature correspond to when $\Gamma(|z|) = K|z|^{\gamma+2}$, where $-3 \leq \gamma \leq 1$. If $0 < \gamma \leq 1$ we say the particles interact through a *hard potential* and if $\gamma < 0$ we say we're dealing with a *soft potential*.

Although not the way it was originally deduced, the (1) equation is generally seen as the asymptotic limit of the Bolzamann equation when the collisions are predominantly grazing, i.e. the angle of deviation in each collision is small. This type of behavior is expected of very hot and rarefied gases and, therefore, the Landau equation finds an important application in plasma physics.

In this case, the particles interact with each other through an *Coulomb* potential, where $s = 1$ and therefore $\gamma = -3$. This is the only physical potential among those dealt with in the literature, which is why it is the one we will focus on in this paper. Despite its physicality, this is also the most singular potential. In fact, we will see later that this singularity imposes problems even in defining a solution to the equation, and we will see that the solutions constructed in the literature for the equation with this potential are of the renormalized type.

The above description only assumes that the particles in question are point-like, as in classical mechanics, and obey Maxwell-Boltzmann type statistics. If, on the other hand, we want a model for quantum particles, some changes must be taken into account.

Take the case of fermions. These are particles that obey the Pauli exclusion principle, which states that no two particles can be in the same quantum state at the same time. Statistically, this implies that the distribution function must have at most one particle per h^3/m^3 of space, implying the pointwise bound

$$(3) \qquad 0 \leq f(t, x, v) \leq \frac{1}{\varepsilon} \equiv \frac{h^3}{m^3 \beta}$$

Above we have that h is the Planck constant, m is the mass of the particle and β is called the degeneracy of the particle. Fermions are particles that undergo a process of *saturation*: if we already have too many particles in a given region of phase space, it becomes less likely that even more particles will enter it. Thus, the collision probabilities for the quantum case are different from those of the classical case, and this new underlying statistics is called Fermi-Dirac. Adapting the Landau equation to these new probabilities gives rise to the so-called Landau-Fermi-Dirac (LFD) equation, which reads

$$(4) \qquad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = \mathrm{div}_v \left(\int a(v - v_*) \big(f_*(1 - \varepsilon f_*) \nabla_v f - f(1 - \varepsilon f) \nabla_{v_*} f_* \big) dv_* \right)$$

Historically, the first adaptation of the kinetic theory of gases to a quantum version began in 1928 with Nordheim [10], followed shortly after by Uehling and Uhlenbeck [12], who adapted the Boltzmann equation to take Fermi-Dirac statistics into account, giving rise to the Boltzmann-Fermi-Dirac equation.

In 1979 Danielewicz [4] used the asymptotic methods used to derive the Landau equation from Boltzmann's to take the limit of the Boltzmann-Fermi-Dirac equation, effectively arriving at the Landau-Fermi-Dirac equation. The same equation, however, had already appeared in the literature back 1970, in the works of Kadomtsev and Pogutse on the relaxation of stellar systems [8].

Once in possession of a quantum version of a classical equation, one wonders about the compatibility between the two. In fact, the correspondence principle of quantum mechanics states that the behavior of quantum systems should be close to that predicted by classical mechanics in the limit of a small Planck constant. In fact, if we formally take the limit $\varepsilon \to 0$ in equation (4), we recover equation (1).

In this article, we propose to show that the above procedure is rigorously justified. That is, that solutions of the Landau-Fermi-Dirac equation converge to solutions of the Landau equation in the limit of a small quantum parameter ε, a procedure mathematically called a *semi-classical limit*. More specifically, we will prove that the LFD solutions constructed in [11] in fact converge to solutions of the Landau equation constructed by Villani in [13], and therefore the Cauchy theories constructed for these two equations are compatible. Furthermore, such a result also justifies the use of the LFD equation as an approximation of the Landau equation.

A result of this type has already been shown for the Boltzmann-Fermi-Dirac equation by He, Lu and Pulvirenti in [7], where it is shown that this equation, in its homogeneous form in space (i.e. the distribution function f depends only on (t, v)), converges to the homogeneous Landau equation. However, semi-classical limit results for the LFD equation are unknown to the author.

2. Preliminaries and main result

Before stating the central result, let's briefly review the Cauchy theory for the Landau and LFD equations. For the Landau equation one can show that formally

$$(5) \qquad \frac{d}{dt} \iint f(t)\varphi \, dxdv = 0$$

if $\varphi = 1, v_i, |v|^2$ or $|x - vt|^2$, which physically corresponds to the conservation of mass, linear momentum, kinetic energy and moment of inertia, respectively. On the other hand, if we define the entropy

$$(6) \qquad H(t) = \iint f \log f \, dxdv,$$

we have that, formally, if f is a solution of (1), then the functional $H(t)$ is decreasing.

Thus, it is reasonable to assume that these quantities are bounded at the time $t = 0$, i.e,

$$(7) \qquad \iint f_0(1 + |x|^2 + |v|^2 + \log f_0) \, dxdv < \infty$$

for a given initial f_0, and so ideally we would like to construct a solution that obeys the conservation laws and entropy decay. In particular, if we had such a solution, then the quantity

$$\sup_{t>0} \iint f(t)(1 + |x|^2 + |v|^2 + \log f(t)) \, dxdv$$

would be bounded by (7), and so we look for our solution in the space of functions such that this quantity is finite.

Here the singularity of the collision kernel comes into play. Indeed, if we consider this kernel for a Coulomb-type potential, the above bounds do not guarantee that the collision integral on the right-hand side of (1) is well-defined, even in the sense of the distributions. One way to get a sense of the solution even with this limitation is to consider *renormalized solutions*.

Formally, the renormalization process is a change in the unknown of the equation. Instead of solving the Landau equation for f, we will choose a smooth, bounded function β and try to solve the equation for $g = \beta(f)$. Since the β function is bounded, ideally we get an unknown g that we know is a bounded function a priori, which can help smooth out the product of singularities.

So let $\beta = \beta(t)$ be a smooth function, called non-linearity. Multiplying the Landau equation by $\beta'(f)$, we get

$$\frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) = \beta'(f) \frac{\partial}{\partial v_i} \left\{ \overline{a}_{ij} \frac{\partial f}{\partial v_j} - \overline{b}_i f \right\},$$

where $\overline{a}_{ij} = a_{ij} *_v f$ and $\overline{b}_i = \frac{\partial \overline{a}_{ij}}{\partial v_j}$. We can rewrite this equation as

$$(8) \quad \frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) = \frac{\partial}{\partial v_i} \left\{ \overline{a}_{ij} \frac{\partial \beta(f)}{\partial v_j} - \overline{b}_i \beta(f) \right\} - \beta''(f) \overline{a}_{ij} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} + \overline{c}[\beta(f) - \beta'(f)f],$$

where $\overline{c} = \frac{\partial^2 \overline{a}_{ij}}{\partial v_i \partial v_j}$, and we say that f is a *renormalized solution* of the Landau equation if the distribution $\beta(f)$ solves (8) in the sense of distributions.

In order to tackle the Coulomb case, we assume throughout the paper that in Landau equation we have the $a(z)$ of the form (2), that the cross-section Γ satisfies

$$(9) \quad \forall R > 0 \text{ there exists a } K_R > 0 \text{ such that } \Gamma(|z|) \geq K_R, \forall |z| \leq R,$$

and that is has the integrability

$$(10) \quad \Gamma(|z|) \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N),$$

for some $r > \frac{N}{N-1}$.

The typical strategy for constructing solutions to this type of equation is to take solutions to a well-chosen approximate equation, whose Cauchy problem is easier to study, and then prove a compactness theorem for the solutions, enabling us to pass this equation to the limit.

For the Landau equation, such a compactness theorem was obtained by Lions in [9]. However, as shown by Villani in [13], in the passage to the limit of the quadratic term in the derivatives $\frac{\partial f}{\partial v_i}$ imposes problems and a defect measure appears, giving rise to an even weaker notion of solution.

Let us then notate $L_2^1(\mathbb{R}_{x,v}^{2N})$ the space of functions g such that

$$\iint_{\mathbb{R}_{x,v}^{2N}} |g(x,v)|(1 + |x|^2 + |v|^2) \, dx dv < \infty.$$

The existence result from [13] then reads

Definition 1. Let $f_0 \in L_2^1(\mathbb{R}_{x,v}^{2N})$ be such that $f_0 \geq 0$ and for every $\delta > 0$, let $\beta_\delta(t) = \frac{t}{1+\delta t}$. We say that $f \in \mathcal{C}((0,\infty); \mathcal{D}'(\mathbb{R}_{x,v}^{2N})) \cap L^\infty((0,\infty); L_2^1(\mathbb{R}_{x,v}^{2N}))$ is a renormalized solution of the Landau equation with defect measure and initial data f_0 if, for every $\delta > 0$, there exists a nonnegative measure μ_δ, bounded on all sets $(0,T) \times \mathbb{R}_x^N \times B_R$, $T < \infty$, such that $\beta_\delta(f)$ satisfies

$$(11) \quad \frac{\partial \beta_\delta(f)}{\partial t} + v \cdot \nabla_x \beta_\delta(f) = \frac{\partial}{\partial v_i} \left\{ \overline{a}_{ij} \frac{\partial \beta_\delta(f)}{\partial v_j} - \overline{b}_i \beta_\delta(f) \right\}$$

$$- \beta''(f) \overline{a}_{ij} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} + \overline{c}[\beta_\delta(f) - \beta_\delta'(f)f] + \mu_\delta$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}_{x,v}^{2N})$, $f(t) \to f_0$ as $t \to 0^+$ in $\mathcal{D}'(\mathbb{R}_{x,v}^{2N})$ and we have

1) Conservation of mass and linear momentum:

$$\iint f(t,x,v)\,dxdv = \iint f_0(x,v)\,dxdv,$$

$$\iint f(t,x,v)v_i\,dxdv = \iint f_0(x,v)v_i\,dxdv \quad \forall i \in \{1,\cdots,N\}.$$

2) Decay of kinetic energy and moment of inertia:

$$\iint f(t,x,v)|v|^2\,dxdv \leq \iint f_0(x,v)|v|^2\,dxdv,$$

$$\iint f(t,x,v)|x-tv|^2\,dxdv \leq \iint f_0(x,v)|x-tv|^2\,dxdv.$$

3) Entropy inequality:

$$\iint f(t,x,v)\log f(t,x,v)\,dxdv + \int_0^t \iint d(\tau,x,v)\,dxdvd\tau \leq \iint f_0(x,v)\log f_0(x,v)\,dxdv,$$

where

$$(12) \qquad d(t,x,v) = \int a(v-v_*)ff_* \left| \frac{\nabla_v f}{f} - \frac{\nabla_{v_*} f_*}{f_*} \right|^{\otimes 2} dv_*,$$

is the entropy dissipation.

More precisely, a renormalized solution with defect measure of the Landau equation satisfies, for every $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}_{x,v}^{2N})$,

$$-\int_{txv} \beta_\delta(f)\frac{\partial \varphi}{\partial t} - \int_{txv} \beta_\delta(f)v \cdot \nabla_x \varphi = \int_{txv} \overline{a}_{ij}\beta_\delta(f)\frac{\partial^2 \varphi}{\partial v_i \partial v_j}$$

$$+ \int_{txv} \left[\frac{\partial \overline{a}_{ij}}{\partial v_j}\beta_\delta(f) + \overline{b}_i\beta_\delta(f) \right] \frac{\partial \varphi}{\partial v_i} + \int_{txv} \overline{a}_{ij}\frac{\partial \gamma_\delta(f)}{\partial v_i}\frac{\partial \gamma_\delta(f)}{\partial v_j}\varphi$$

$$+ \int_{txv} \overline{c}[\beta_\delta(f) - f\beta_\delta'(f)]\varphi + \int_{txv} \varphi d\mu$$

and we also have, for every $\psi \in \mathcal{D}(\mathbb{R}_x^N \times \mathbb{R}_v^N)$,

$$\int_{xv} f(t)\psi \xrightarrow{t\to 0^+} \int_{xv} f_0\psi.$$

Some nuances of the above definition are worth highlighting. Note that while the equation on $(0,\infty) \times \mathbb{R}_{x,v}^{2N}$ is stated for $\beta_\delta(f)$, the statement of the initial condition rests solely on the function f, without renormalization. In addition, some terms in the above statement are not well defined or do not have a canonical definition.

Remark 1. In equation (11), since we have no regularity for the derivatives of f, the term $\overline{a}_{ij}\frac{\partial \beta_\delta(f)}{\partial v_j}$ must be interpreted as $\frac{\partial}{\partial v_j}[\overline{a}_{ij}\beta_\delta(f)] - \frac{\partial \overline{a}_{ij}}{\partial v_j}\beta_\delta(f)$, in the sense of distributions.

Remark 2. Since we don't suppose any regularity upon renormalized solutions, one might ask what the definition of the quadratic term $-\beta''(f)\overline{a}_{ij}\frac{\partial f}{\partial v_i}\frac{\partial f}{\partial v_j}$ in the sense of distributions is. In fact, this should be viewed as a notation for

$$(13) \qquad \int_{v_*} \left| \sqrt{a(v-v_*)}\sqrt{f_*}\nabla_v \gamma_\delta(f) \right|^2,$$

where inside the square we have an $L^2_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ function, which is defined in the sense of distributions as

$$\sqrt{a(v-v_*)}\sqrt{f_*}\nabla_v \gamma_\delta(f) = \mathrm{div}_v \left[\sqrt{a(v-v_*)}\sqrt{f_*}\gamma_\delta(f)\right] - \mathrm{div}_v \left[\sqrt{a(v-v_*)}\right]\sqrt{f_*}\gamma_\delta(f)$$

This way, the term $-\beta''(f)\overline{a}_{ij}\frac{\partial f}{\partial v_i}\frac{\partial f}{\partial v_j}$ is an $L^1((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ function, independent of the sequence of approximating functions $(f^n)_n$. However, we cannot interpret this immediately as a pointwise product of functions, and this term should rather be seen as a "black box".

Remark 3. In the same vein as the definition of the quadratic term above, the entropy dissipation (12) should be interpreted as the $L^2((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ norm of the function

$$\mathrm{div}_v \left(\sqrt{a(v-v_*)}\sqrt{f}\sqrt{f_*}\right) - \mathrm{div}_v \left(\sqrt{a(v-v_*)}\right)\sqrt{f}\sqrt{f_*}$$
$$- \mathrm{div}_{v_*} \left(\sqrt{a(v-v_*)}\sqrt{f}\sqrt{f_*}\right) + \mathrm{div}_{v_*} \left(\sqrt{a(v-v_*)}\right)\sqrt{f}\sqrt{f_*}.$$

For the Landau-Fermi-Dirac (LFD) equation, we have the same conservation laws (5) of mass, momentum, kinetic energy and moment of inertia. However, the saturation effect described by the Pauli exclusion principle leads us to consider, instead of (6), the *quantum* entropy

$$(14) \qquad\qquad S_\varepsilon(g) = -\frac{1}{\varepsilon}\int_{xv} \varepsilon g \log(\varepsilon g) + (1 - \varepsilon g)\log(1 - \varepsilon g),$$

which is a decreasing functional in the solutions of the Landau-Fermi-Dirac equation (4).

For the quantum case, the extra bound (3) implies that the collision kernel is well defined even in the case of a Coulomb-type potential. Thus, we don't need renomalization to show the existence of a global solution and we can construct a weak solution, in the sense of distributions, without any problems.

This way, the existence result from [11] reads

Definition 2. Let $f_0 \in L^1_2(\mathbb{R}^{2N}_{x,v})$ be such that $0 \leq f_0 \leq \varepsilon^{-1}$. A function $f = f(t,x,v)$ in $C((0,\infty); \mathcal{D}'(\mathbb{R}^{2N}_{x,v})) \cap L^\infty((0,\infty); L^1_2(\mathbb{R}^{2N}_{x,v}))$ is called a global weak solution of LFD with initial data f_0 and quantum parameter $\varepsilon > 0$ if it satisfies

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{\partial^2(\overline{a}_{ij}f)}{\partial v_i \partial v_j} + \frac{\partial}{\partial v_i}\left[\frac{\partial \overline{a}_{ij}}{\partial v_j}f + \overline{b}_i f(1 - \varepsilon f)\right]$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{2N}_{x,v})$, $f(t) \to f_0$ in $\mathcal{D}'(\mathbb{R}^{2N}_{x,v})$ and moreover

1) Pauli exclusion principle:
$$0 \leq f(t) \leq \varepsilon^{-1}.$$

2) Conservation of mass and linear momentum:
$$\iint f(t,x,v)\,dxdv = \iint f_0(x,v)\,dxdv,$$
$$\iint f(t,x,v)v_i\,dxdv = \iint f_0(x,v)v_i\,dxdv \quad \forall i \in \{1,\cdots,N\}.$$

3) Decay of kinetic energy and moment of inertia:
$$\iint f(t,x,v)|v|^2\,dxdv \leq \iint f_0(x,v)|v|^2\,dxdv,$$
$$\iint f(t,x,v)|x-tv|^2\,dxdv \leq \iint f_0(x,v)|x-tv|^2\,dxdv.$$

4) Entropy inequality:

$$\iint s(t,x,v)\,dxdv + \int_0^t \iint d(\tau,x,v)\,dxdvd\tau \le \iint s(0,x,v)\,dxdv,$$

where

$$s(t,x,v) = \varepsilon f \log(\varepsilon f) + (1-\varepsilon f)\log(1-\varepsilon f)$$

is the quantum entropy and

(15)
$$d(t,x,v) = \int a(v-v_*)f(1-\varepsilon f)f_*(1-\varepsilon f_*)\left| \frac{\nabla_v f}{f(1-\varepsilon f)} - \frac{\nabla_{v_*} f_*}{f_*(1-\varepsilon f_*)} \right|^{\otimes 2} dv_*,$$

is the quantum entropy dissipation.

More precisely, f is a global weak solution of LFD if for every test function $\varphi \in \mathcal{D}((0,\infty) \times \mathbb{R}_x^N \times \mathbb{R}_v^N))$, we have

$$\int_{txv} f\frac{\partial \varphi}{\partial t} + \int_{txv} fv_i\frac{\partial \varphi}{\partial x_i} = -\int_{txv} \bar{a}_{ij}f\frac{\partial^2 \varphi}{\partial v_i \partial v_j} - \int_{txv}\left[\frac{\partial \bar{a}_{ij}}{\partial v_j}f + \bar{b}_i f(1-\varepsilon f)\right]\frac{\partial \varphi}{\partial v_i}$$

and for every $\psi \in \mathcal{D}(\mathbb{R}_x^N \times \mathbb{R}_v^N)$, we have

$$\int_{xv} f(t)\psi \xrightarrow{t\to 0^+} \int_{xv} f_0\psi.$$

Remark 4. As is the case for the classical Landau equation, here the lack of regularity a priori in v implies that one should interpret the entropy dissipation (15) in a different way. The result in [11] proves that the entropy inequality is valid is we interpret the integral $\int_0^t \iint d(\tau,x,v)\,dxdvd\tau$ as a notation for the $L^2((0,t)\times\mathbb{R}_x^N\times\mathbb{R}_v^N\times\mathbb{R}_{v_*}^N)$ norm of

$$\mathrm{div}_v\left[\sqrt{a(v-v_*)}\sqrt{f_*(1-f_*)}\arcsin\sqrt{f}\right] - \mathrm{div}_{v_*}\left[\sqrt{a(v-v_*)}\sqrt{f(1-f)}\arcsin\sqrt{f_*}\right]$$

$$- \mathrm{div}_v\left(\sqrt{a(v-v_*)}\right)\sqrt{f_*(1-f_*)}\arcsin\sqrt{f} + \mathrm{div}_{v_*}\left(\sqrt{a(v-v_*)}\right)\sqrt{f(1-f)}\arcsin\sqrt{f_*},$$

where the divergence of a matrix A is defined as $\mathrm{div}_v A = \sum_j \frac{\partial A_{ij}}{\partial v_j}$. The motivation for using the expression (15) is that if f has some regularity in v, then the two expressions are equal.

The existence of such solutions was proven in [11] by a compactness argument reminiscent of the work of Lions and Villani for the Landau equation. In this approach, we construct a sequence of equations that adequately approximates the LFD equation and then deduce a compactness theorem for its solutions, which allows us to pass to the limit in the approximations and obtain a solution to the LFD equation.

The technique we will use to prove a semi-classical limit for solutions of LFD depends on being able to pass from a weak formulation of this equation to a renormalized one, and therefore to manipulate products of derivatives of these solutions. We don't have regularity results for these solutions, but it turns out that it's enough to consider solutions that originate from solutions to approximate, more regular equations, which leads us to the definition of *suitable* weak solutions of the LFD equation.

Definition 3. A weak solution f to the LFD is called a *suitable weak solution* if there exists a sequence $f_m \in L^2((0,T)\times\mathbb{R}_x^N; H^1(\mathbb{R}_v^N))$ of solutions to

(16)
$$\begin{cases} \dfrac{\partial f_m}{\partial t} + v\cdot\nabla_x f_m = \dfrac{\partial}{\partial v_j}\left[\bar{a}_{ij}^m\dfrac{\partial f_m}{\partial v_i} - \bar{b}_j^m f_m(1-\varepsilon f_m)\right] \\ f_m|_{t=0} = f_{0,m} \end{cases}$$

such that $f_m \to f$ almost everywhere in $(0,\infty)\times\mathbb{R}_{x,v}^{2N}$ and

i. Convergence of the coefficients: $\overline{a}^m, \operatorname{div}_v \overline{a}^m, \overline{b}^m, \operatorname{div}_v \overline{b}^m$ are $L^\infty_{loc}((0,T) \times \mathbb{R}^{2N}_{x,v})$ functions such that

$$\overline{a}^m \to \overline{a} \qquad\qquad \overline{b}^m \to \overline{b}$$

$$\operatorname{div}_v \overline{a}^m \to \operatorname{div}_v \overline{a} \qquad \operatorname{div}_v \overline{b}^m \to \operatorname{div}_v \overline{b}$$

in $L^1_{loc}((0,T) \times \mathbb{R}^{2N}_{x,v})$.

ii. Uniform bound: There exists a $C > 0$ such that, for every $m \in \mathbb{N}$,

$$(17) \qquad \int_0^T \iint_{\mathbb{R}^{2N}_{x,v}} f_m (1 + |v|^2 + |x - tv|^2)\, dxdvdt \le C \text{ and } 0 \le f_m \le \varepsilon^{-1}.$$

We call the sequence of approximating functions $(f_m)_m$ an *approximating scheme*. The matrix $\overline{a}^m$ is called the diffusion matrix and it is sometimes useful to explicitly state which sequence of diffusion matrices we're using. In this case, we say that $f_m \to f$ is an approximation scheme with diffusion $(\overline{a}^m)_m$.

Note that, by Vitali's convergence theorem, the bounds (17) imply that the convergence $f_m \to f$ holds in $L^1_{loc}((0,\infty); L^1(\mathbb{R}^{2N}_{x,v}))$. Also, the solutions constructed in [11] are *suitable* weak solutions and if we had uniqueness for the weak solutions of LFD then all weak solutions are suitable.

The main theorem reads as follows

Theorem 1. *Let $\varepsilon_n \to 0$. For each n, let f^n be a suitable solution of the LFD equation with quantum parameter ε_n and initial data $0 \le f_0^n \le \varepsilon_n^{-1}$. For each n, let $f_m^n \to f^n$ be an approximating scheme with diffusion $(\overline{a}^{n,m})_m$.*

Suppose the diffusion matrices satisfy

$$(18) \qquad \overline{a}^{n,m} \ge a_m *_v (f_m^n (1 - \varepsilon f_m^n)), \text{ where } a_m(z) = \Gamma_m(|z|)\left(I - \frac{z \otimes z}{|z|^2}\right),$$

with Γ_m such that $\Gamma_m(|z|) \to \Gamma(|z|)$ in $L^1_{loc}(\mathbb{R}^N)$ and for every $R > 0$ there exists some $K_R > 0$ such that

$$\Gamma_m(|z|) \ge K_R \quad \forall |z| \le R.$$

If the sequence of initial data $(f_0^n)_n$ is such that

$$(19) \qquad \iint_{\mathbb{R}^{2N}_{x,v}} f_0^n (1 + |x|^2 + |v|^2 + \log f_0^n)\, dxdv \to \iint_{\mathbb{R}^{2N}_{x,v}} f_0 (1 + |x|^2 + |v|^2 + \log f_0)\, dxdv$$

then, up to a subsequence, the sequence f^n converges to a renormalized solution of Landau equation with defect measure.

Hypothesis (18) arises as a requirement of the techniques we use and also as a result of the fact that we would like to write the quadratic term of the renormalized Landau form in the "natural" form (13). However, this convolution structure is not essential for demonstrating most of the results and if we are willing to offer an alternative, albeit more abstract, interpretation for the quadratic term, we can alleviate the requirement (18), which specifies the way in which we approximate the diffusion matrix $\overline{a}$, in favor of a slightly more general ellipticity requirement.

Theorem 2. *Let $\varepsilon_n \to 0$. For each ε_n, let f^n be a suitable solution of the LFD equation with quantum parameter ε_n and initial data f_0^n. For each n, let $f_m^n \to f^n$ be an approximating scheme with diffusion $(\overline{a}^{n,m})_m$.*

If the diffusion matrices are such that $\frac{\partial \overline{a}^{n,m}}{\partial v_k} \in L^\infty_{loc}((0,T) \times \mathbb{R}^{2N}_{x,v})$ converges to $\frac{\partial \overline{a}^n}{\partial v_k}$ in $L^1_{loc}((0,T) \times \mathbb{R}^{2N}_{x,v})$, for every $i,j,k = 1,\ldots,N$ and such that for every $R,R' > 0$ and $\eta \in \mathbb{R}^N$,

$$(20) \qquad \overline{a}^{n,m}_{ij} \eta_i \eta_j \geq \int_{|v_*| \leq R'} \nu \left[1 - \left(\frac{v - v_*}{|v - v_*|} \cdot \frac{\eta}{|\eta|} \right)^2 \right] |\eta|^2 f^n_{m,*}(1 - f^n_{m,*}) \, dv_*, \ \forall |v| \leq R,$$

then we can write the quadratic term in Landau equation as

$$-\beta''(f)\overline{a}_{ij} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} = \liminf_{k \to \infty} \left| S^k(\overline{a}) \nabla_v \gamma(f) \right|^2,$$
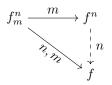
where the term inside the square is an $L^2_{loc}((0,T) \times \mathbb{R}^{2N}_{x,v})$ vector function, defined in the sense of distributions as

$$S^k(\overline{a}) \nabla_v \gamma(f) \equiv \mathrm{div}_v[S^k(\overline{a})\gamma(f)] - \mathrm{div}_v[S^k(\overline{a})]\gamma(f),$$

and S^k is some smooth approximation of the matrix square root.

We remark that the statement (20), although very close, is a weaker requirement than (18).

The technique we will use to prove Theorems 1 and 2 consists of using the extra regularity provided by the $(f^n_m)_n$ approximations to pass the equation into a renormalized form and then study the compactness of the diagonal of the scheme

$$
\begin{array}{ccc}
f^n_m & \xrightarrow{\ \ m\ \ } & f^n \\
& \searrow{\scriptstyle n,m} & \Big\downarrow{\scriptstyle n} \\
& & f
\end{array}
$$

(21)

That is, show that there exists a sequence $m_n \to \infty$ such that $(f^n_{m_n})_n$ is strongly compact in L^1. Since the approximation schemes (f^n_m) are always "close" to f^n, we can then show the convergence of f^n "in tow" to the same limit, which we will show is a renormalized solution of the Landau equation with defect measure.

In order to do this, in Section 3 we start by proving that any time we have such an arrangement, there exists some diagonal that is weakly compact. We then proceed in Section 4 to show that approximate solutions satisfy in fact a renormalized formulation of LFD, which we will use in Section 5 to prove that our diagonal is in fact strongly compact. Section 6 is then dedicated to using this compactness to pass the renormalized formulation in the limit, thus proving Theorem 1, and finally in Section 7 we prove Theorem 2 by studying approximations of the matrix square root.

3. Weak compactness of diagonal sequences

In the existence theory for the Boltzmann, Landau and Landau-Fermi-Dirac equations, the conservation laws obeyed by these equations automatically imply their solutions are weakly compact in $L^1_{loc}((0,\infty), L^1(\mathbb{R}^{2N}_{x,v}))$.

In our case, however, the approach needs to be slightly different, since our approximation schemes don't necessarily satisfy conservation laws. In fact, this is neither assumed about the schemes nor can it be deduced from the approximating equations themselves, since we don't even have the convolution structure of $\overline{a}$, $\overline{b}$ and $\overline{c}$ for these equations. Despite this, we can construct a diagonal sequence that is weakly compact in L^1, and the main result of this section then reads

Lemma 1. *Let f^n be a sequence of suitable weak solutions to the LFD equation (4), as in Definition 3 and for each n, consider $f^n_m \to f^n$ an approximation scheme for f^n.*

If there exists a $C > 0$ such that

$$\iint_{\mathbb{R}^{2N}_{x,v}} f_0^n (1 + |x|^2 + |v|^2 + \log f_0^n) \, dx dv, \leq C$$

then there exists a sequence $(M_n)_n$ such that if $m_n \geq M_n$ for every $n \in \mathbb{N}$, then the diagonal sequence $(f_{m_n}^n)_n$ is weakly compact in $L^1_{loc}((0,\infty); L^1(\mathbb{R}^{2N}_{x,v}))$.

Proof. The main idea is that although we don't know if the approximate solutions obey conservation laws, the f^n solutions definitely do, so we choose a diagonal sequence $f_{m_n}^n$ that is close enough to the f^n sequence.

Note that the uniform bound implies weak compactness in L^1 by a simple application of the Dunford-Pettis theorem.

Part I: Bounding the approximating terms by their limits

We'll start by showing that for every $T > 0$ and $n \in \mathbb{N}$ there is a $C_T > 0$ and an $M_{n,T} > 0$ such that

$$(22) \qquad \int_0^T \int_{xv} f_m^n (1 + |v| + |x - tv| + |\log f_m^n|) \leq C_T + \int_0^T \int_{xv} f^n \log f^n$$

for every $m \geq M_{n,T}$.

Indeed for every n the sequence $(f_m^n(1+|v|^2+|x-tv|^2))_m$ is uniformly bounded in $L^1((0,T); L^1(\mathbb{R}^{2N}_{x,v}))$ and $f_m^n \to f^n$ in $L^1((0,T); L^1(\mathbb{R}^{2N}_{x,v}))$, hence it follows by interpolation that

$$(23) \qquad f_m^n(1 + |v| + |x - tv|) \to f^n(1 + |v| + |x - tv|) \text{ in } L^1((0,T); L^1(\mathbb{R}^{2N}_{x,v})),$$

thus there exists some $M_{n,T} > 0$ such that

$$\int_0^T \int_{xv} f_m^n (1 + |v| + |x - tv|) \leq \int_0^T \int_{xv} f^n(1 + |v| + |x - tv|) + 1$$

$$\leq 2 \int_0^T \int_{xv} f^n(1 + |v|^2 + |x - tv|^2) + 1.$$

for every $m \geq M_{n,T}$. Then, since f^n is a solution of (4) as in Definition 2, we have that

$$\int_0^T \int_{xv} f^n(1 + |v|^2 + |x - tv|^2) \leq T \int_{xv} f_0^n(1 + |v|^2 + |x|^2),$$

which then achieves the first part of inequality (22).

For the entropy part notice that, for any function $h = h(t,x,v)$ we have

$$h|\log h| = h \log h - 2h \log h \mathbb{1}_{\{h \leq 1\}}$$

We can bound the last term as

$$-h \log h \mathbb{1}_{\{h \leq 1\}} = h \log(1/h) \mathbb{1}_{\{h \leq \exp(-|x|-|v|)\}} + h \log(1/h) \mathbb{1}_{\{h > \exp(-|x|-|v|)\}}$$
$$\leq h \log(1/h) \mathbb{1}_{\{h \leq \exp(-|x|-|v|)\}} + h(|x| + |v|)$$

and since $x \log(1/x) \leq \sqrt{x}$, for every $x \geq 0$, we conclude that

$$-h \log h \mathbb{1}_{\{h \leq 1\}} \leq \exp\left(-(|x| + |v|)/2\right) + h(|x| + |v|),$$

thus

$$(24) \qquad h|\log h| \leq 2 \exp\left[-\frac{1}{2}(|x| + |v|)\right] + h \log h + 2h(|x| + |v|).$$

Then choosing $h(t, x', v) = f^n(t, x' + tv, v)$ and letting $x = x' + tv$ we have that

$$(25) \quad f^n(t, x, v)|\log f^n(t, x, v)| \leq 2 \exp\left[-\frac{1}{2}\big(|x - tv| + |v|\big)\right] + f^n(t, x, v)\log f^n(t, x, v)$$
$$+ 2f^n(t, x, v)\big(|x - tv| + |v|\big).$$

for every $(t, x, v) \in (0, T) \times \mathbb{R}^{2N}_{x,v}$. So in particular, for $0 \leq f^n_m \leq \varepsilon_n^{-1}$,

$$f^n_m|\log f^n_m| \leq 2\exp\left[-\frac{1}{2}\big(|x - tv| + |v|\big)\right] + (2 + \varepsilon_n^{-1})f^n_m\big(1 + |x - tv| + |v|\big).$$

hence from (23) it follows that $f^n_m|\log f^n_m| \to f^n|\log f^n|$ in $L^1((0, T); L^1(\mathbb{R}^{2N}_{x,v}))$, for each n by dominated convergence. There exists some M_T such that

$$\int_0^T \int_{xv} f^n_m|\log f^n_m| \leq \int_0^T \int_{xv} f^n|\log f^n| + 1$$

for every $m \geq M_{n,T}$. Then, applying inequality (25) leads

$$\int_0^T \int_{xv} f^n_m|\log f^n_m| \leq C_N + C_m \int_0^T \int_{xv} f^n \log f^n$$

which implies the inequality that we wanted.

Now, since f^n is a solution of (4) as in Definition 2, we have that

$$\int_0^T \int_{xv} f^n(1 + |v|^2 + |x - tv|^2) \leq T \int_{xv} f^n_0(1 + |v|^2 + |x|^2),$$

which then proves inequality (22).

Part II: Entropy bound

We now pass to the entropy part of the proposition. A remarkable property of the functionals (14) is that although the LFD equation (4) formally becomes the Landau equation (1) in the $\varepsilon \to 0$ limit, we don't have formal convergence from the quantum entropy to the classical entropy.

Indeed, any function of the form $f^\varepsilon(t, x, v) = \varepsilon^{-1}\mathbb{1}_A(x - tv, v)$ is a weak solution to (4) that conserves mass and kinetic energy and is such that $S_\varepsilon(f^\varepsilon) = 0$. But notice that $H(f^\varepsilon) \to \infty$, which implies that we cannot control the classical entropy of a semi-classical limit just by an uniform control over the quantum entropies.

Thus, we see that not every semi-classical limit of LFD solutions gives rise to a physical result with only a uniform estimate on the quantum entropy, and another estimate must then be found. Let us then show that

$$(26) \qquad\qquad \int_{xv} f^n \log f^n \leq \int_{xv} f^n_0 \log f^n_0 + \int_{xv} f^n_0.$$

Since $0 \leq f^n_0 \leq \varepsilon^{-1}$, we have that

$$\int_{xv} f^n_0 \log f^n_0 \geq \frac{1}{\varepsilon} \int_{xv} \varepsilon f^n_0 \log f^n_0 + (1 - \varepsilon f^n_0)\log(1 - \varepsilon f^n_0)$$

$$= S_\varepsilon(f^n_0) - \int_{xv} f^n_0 \log(\varepsilon)$$

$$\geq S_\varepsilon(f(t)) - \int_{xv} f^n_0 \log(\varepsilon)$$

Now, using that for every $0 \leq x \leq \varepsilon^{-1}$, we have

$$\frac{1}{\varepsilon} \left[\varepsilon x \log(\varepsilon x) + (1 - \varepsilon x) \log(1 - \varepsilon x) \right] \geq x \log x + x \log \varepsilon - x,$$

it follows that

$$\int_{xv} f_0^n \log f_0^n \geq \int_{xv} f^n \log f^n + \int_{xv} f^n \log(\varepsilon) - \int_{xv} f^n - \int_{xv} f_0^n \log(\varepsilon),$$

and then (26) follows from the conservation of mass of f^n.

Part III: Weak compactness

Now, applying (22) together with (26) we have that for all $T > 0$ and $n \in \mathbb{N}$ there exists a $C_T > 0$ and an $M_{n,T} > 0$ such that

$$\int_0^T \int_{xv} f_m^n (1 + |v| + |x - tv| + |\log f_m^n|) \leq C_T$$

for all $m \geq M_{n,T}$, which is indeed the first part of the statement. In this final step we show that the bound we have shown indeed implies the weak compactness from the statement.

Let $N_n = \max_{T \in \{1,\ldots,n\}} M_{n,T}$ and suppose that the diagonal sequence $(f_{m_n}^n)_n$ satisfies $m_n \geq N_n$ for every $n \in \mathbb{N}$. For every $n \geq 1$ we have that $m_n \geq N_n \geq M_{n,1}$ hence

$$(27) \qquad \int_0^1 \int_{xv} f_{m_n}^n (1 + |v|^2 + |x - tv|^2 + |\log f_{m_n}^n|) \leq C_1.$$

Let us write g^n instead of $f_{m_n}^n$. The bound (27) gives us, from the Dunford-Pettis theorem, that there exists a sequence $k_1(n)$ such that

$$g^{k_1(n)} \rightharpoonup g^1 \text{ in } L^1((0,1) \times \mathbb{R}_{x,v}^{2N}).$$

Next, for each $n \geq 2$ we have that $m_n \geq N_n \geq M_{n,2}$ and hence

$$\int_0^2 \int_{xv} f_{m_n}^n (1 + |v|^2 + |x - tv|^2 + |\log f_{m_n}^n|) \leq C_2$$

and similarly this implies there exists a $k_2(n)$, subsequence of $k_1(n)$, such that

$$g^{k_2(n)} \rightharpoonup g^2 \text{ in } L^1((0,2) \times \mathbb{R}_{x,v}^{2N})$$

and from uniqueness of the weak limit, we have that $g^2 = g^1$ a.e. in $(0,1) \times \mathbb{R}_{x,v}^{2N}$.

We repeat this argument, taking subsequences of subsequences to construct, for every m, a sequence $(g^{k_m(n)})_n$ that converges weakly in $L^1((0,m) \times \mathbb{R}_{x,v}^{2N})$ to g^m. As before, we have that if $N < M$ then g^N and g^M coincide in $t \in (0, N)$. Thus, if we consider

$$g = \sum_{n=1}^{\infty} g^n \mathbb{1}_{[n-1,n)}(t),$$

then g^N is the restriction of g to $(0, N)$.

The diagonal sequence $g^{k_n(n)}$ converges to g weakly in $L^1_{loc}((0,\infty); L^1(\mathbb{R}_{x,v}^{2N}))$. Indeed, for every $T > 0$, let $N > T$. By construction, the sequence $g^{k_n(n)}$ is a subsequence of $g^{k_N(n)}$, for $n > N$, thus it converges to g^N in $(0, N)$ and in particular it converges to g in $(0, T)$.

We then relabel the diagonal sequence $k_n(n)$ simply as n and we have, to summarize,

$$f_{m_n}^n \rightharpoonup g \text{ in } L^1_{loc}((0,\infty); L^1(\mathbb{R}_{x,v}^{2N})).$$

$\square$

4. From weak to renormalized solutions

A crucial step in the proof of Theorem 1 consists to find a renormalized formulation for the LFD equation, which we will later pass to the limit by making the quantum parameter tend to zero. So far it is not known whether weak solutions of the LFD are also renormalized solutions (and in what sense these renormalized solutions would be defined), so in this section we will show that if we had a little more regularity in the variable v, then weak solutions are renormalized and vice versa.

Proposition 1. *Suppose* $g \in L^2_{loc}((0,\infty) \times \mathbb{R}^N_x; H^1_{loc}(\mathbb{R}^N_v)) \cap L^\infty((0,\infty) \times \mathbb{R}^{2N}_{x,v})$, *let* $\beta_\delta(t) \equiv \frac{t}{1+\delta t}$ *and* $\overline{A}_{ij}, \overline{B}_i, \frac{\partial \overline{B}_i}{\partial v_i} \in L^\infty_{loc}$. *Thus,* g *is a solution to*

$$
(28) \qquad \frac{\partial g}{\partial t} + v_i \frac{\partial g}{\partial x_i} = \frac{\partial}{\partial v_j} \left\{ \overline{A}_{ij} \frac{\partial g}{\partial v_i} - \overline{B}_i g(1 - \varepsilon g) \right\}
$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ *if and only if* $\beta_\delta(g)$ *is a solution to*

$$
(29) \quad \frac{\partial \beta_\delta(g)}{\partial t} + v \cdot \nabla_x \beta_\delta(g) = \frac{\partial}{\partial v_i} \left\{ \overline{A}_{ij} \frac{\partial \beta_\delta(g)}{\partial v_j} - \overline{B}_i (1 - 2\varepsilon g)\beta_\delta(g) - 2\varepsilon \overline{B}_i B_\delta(g) \right\}
$$
$$
- \beta''_\delta(g) \overline{A}_{ij} \frac{\partial g}{\partial v_i} \frac{\partial g}{\partial v_j} - \overline{C} \left[g(1 - \varepsilon g)\beta'_\delta(g) - (1 - 2\varepsilon g)\beta_\delta(g) - 2\varepsilon B_\delta(g) \right]
$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{2N}_{x,v})$, *where* $\overline{C} = \sum_i \frac{\partial \overline{B}_i}{\partial v_i}$ *and* $B_\delta(x) = \int_0^x \beta_\delta(t)\, dt$.

Proof. We begin with the direct implication. The formal procedure consists of multiplying both sides of the equation by β'_δ and then rewriting the derivatives in the appropriate form, using the product rule. However, due to the low regularity of g (particularly with respect to the variables t and x), we cannot justify this procedure directly.

To overcome this, take ρ to be a $C^\infty(\mathbb{R}^{2N}_{x,v})$ function with support in $(-1,0) \times B_x(0,1)$ and define the mollifying sequence

$$
\rho_m(t,x) = \frac{1}{m^{N+1}} \rho\left(\frac{t}{m}, \frac{x}{m} \right).
$$

Fix $t > 0$ and $x \in \mathbb{R}^N_x$ and consider a test function $\varphi = \varphi(v)$ in $C^\infty_c(\mathbb{R}^N_v)$. Testing equation (28) against

$$
\Psi_{t,x} = \begin{cases} \mathbb{R}_t \times \mathbb{R}^N_x & \to \mathbb{R} \\ (s,y) & \mapsto \rho_m(t-s, x-y)\varphi(v) \end{cases}
$$

leads

$$
- \int_0^\infty \iint g(s,y,v) \left(\frac{\partial \rho_m}{\partial t} + v_i \frac{\partial \rho_m}{\partial x_i} \right)(t-s, x-y, v)\varphi(v)\, dy\, dv\, ds
$$
$$
= \int_0^\infty \iint \left[\overline{A}_{ij} \frac{\partial g}{\partial v_j} - \overline{B}_i g(1 - \varepsilon g) \right](s,y,v) \rho_m(t-s, x-y) \frac{\partial \varphi}{\partial v_i}(v)\, dy\, dv\, ds,
$$

which are convolutions (in the variables t and x) with ρ_m. Here it will be useful to establish a notation for the rest of the proof. For any distribution T, let's note T_m the convolution

$$
T_m \equiv T *_{t,x} \rho_m.
$$

This way, we rewrite the above expression as

$$-\int \frac{\partial g_m}{\partial t}(t,x,v)\varphi(v)\, dv - \int v_i \frac{\partial g_m}{\partial x_i}(t,x,v)\varphi(v)\, dv$$

$$= \int \left[\overline{A}_{ij}\frac{\partial g}{\partial v_j} - \overline{B}_i g(1-\varepsilon g)\right]_m (t,x,v)\frac{\partial\varphi}{\partial v_i}(v)\, dv,$$

for every $t > 0$ and $x \in \mathbb{R}_x^N$.

Let $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R}_{x,v}^{2N})$, taking $v \mapsto \varphi(t,x,v)$ as a test function in the expression above and integrating in $(t,x) \in (0,+\infty)\times\mathbb{R}^N$ we obtain

$$-\int_{txv}\frac{\partial g_m}{\partial t}\varphi - \int_{txv}v_i\frac{\partial g_m}{\partial x_i}\varphi = \int_{txv}\left[\overline{A}_{ij}\frac{\partial g}{\partial v_j} - \overline{B}_i g(1-\varepsilon g)\right]_m\frac{\partial\varphi}{\partial v_i},$$

where we have used the notation $\int_{txv}$ to denote $\int_0^\infty\int_{xv}\, dt$.

Now let χ_k be a mollifying sequence in all three variables (t,x,v), with support in $(-1,0)\times B_x(0,1)\times B_v(0,1)$. Taking $(\beta'(g_m)*\chi_k)\varphi$ as a test function in the equation above, we have

$$(30) \quad -\int_{txv}\frac{\partial g_m}{\partial t}(\beta'(g_m)*\chi_k)\varphi - \int_{txv}v_i\frac{\partial g_m}{\partial x_i}(\beta'(g_m)*\chi_k)\varphi$$

$$= \int_{txv}\left[\overline{A}_{ij}\frac{\partial g}{\partial v_j} - \overline{B}_i g(1-\varepsilon g)\right]_m\frac{\partial[(\beta'(g_m)*\chi_k)\varphi]}{\partial v_i}.$$

We start by passing this equation to the limit $k\to\infty$. To lighten the notation, for the rest of this proof, let us simply note L_{loc}^p for $L_{loc}^p((0,\infty)\times\mathbb{R}_{x,v}^{2N})$. Since $|\beta'(g_m)| \le 1$, we have, by standard convolution results,

$$\frac{\partial g_m}{\partial t}(\beta'(g_m)*\chi_k) \xrightarrow{k\to\infty} \frac{\partial g_m}{\partial t}\beta'(g_m) = \frac{\partial\beta(g_m)}{\partial t}$$

$$v_i\frac{\partial g_m}{\partial x_i}(\beta'(g_m)*\chi_k) \xrightarrow{k\to\infty} v_i\frac{\partial g_m}{\partial x_i}\beta'(g_m) = v_i\frac{\partial\beta(g_m)}{\partial x_i}$$

in L_{loc}^p, for every $1 \le p < \infty$, hence the left-hand side of (30) converges to

$$-\int_{txv}\frac{\partial\beta(g_m)}{\partial t}\varphi - \int_{txv}v_i\frac{\partial\beta(g_m)}{\partial x_i}\varphi = \int_{txv}\beta(g_m)\frac{\partial\varphi}{\partial t} + \int_{txv}\beta(g_m)v_i\frac{\partial\varphi}{\partial x_i},$$

after integration by parts. For the right-hand side of (30), consider

$$\frac{\partial[(\beta'(g_m)*\chi_k)\varphi]}{\partial v_i} = \left[\left(\beta''(g_m)\frac{\partial g_m}{\partial v_i}\right)*\chi_k\right]\varphi + (\beta'(g_m)*\chi_k)\frac{\partial\varphi}{\partial v_i}.$$

Using that $\beta''(g_m)\frac{\partial g_m}{\partial v_i}$ and $\beta'(g_m)$ are in L_{loc}^2, the above expression converges to

$$\beta''(g_m)\frac{\partial g_m}{\partial v_i}\varphi + \beta'(g_m)\frac{\partial\varphi}{\partial v_i}$$

in L_{loc}^2 and is compactly supported. Since $\left[\overline{A}_{ij}\frac{\partial g}{\partial v_j} - \overline{B}_i g(1-\varepsilon g)\right]_m \in L_{loc}^2$, passing (30) to the limit $k\to\infty$ gives

$$(31) \quad \int_{txv}\beta(g_m)\frac{\partial\varphi}{\partial t} + \int_{txv}\beta(g_m)v_i\frac{\partial\varphi}{\partial x_i}$$

$$= \int_{txv}\left[\overline{A}_{ij}\frac{\partial g}{\partial v_j} - \overline{B}_i g(1-\varepsilon g)\right]_m\left(\beta''(g_m)\frac{\partial g_m}{\partial v_i}\varphi + \beta'(g_m)\frac{\partial\varphi}{\partial v_i}\right),$$

and it remains to pass this equation to the limit $m\to\infty$.

The left-hand side of (31) converges to

$$\int_{txv} \beta(g)\frac{\partial\varphi}{\partial t} + \int_{txv} \beta(g)v_i\frac{\partial\varphi}{\partial x_i}$$

since $g_m \to g$ almost everywhere and $|\beta(g_m)| \leq C$. For the integral

$$\int_{txv} \left[\overline{A}_{ij}\frac{\partial g}{\partial v_j}\right]_m \left(\beta''(g_m)\frac{\partial g_m}{\partial v_i}\varphi + \beta'(g_m)\frac{\partial\varphi}{\partial v_i}\right),$$

we have that $\left[\overline{A}_{ij}\frac{\partial g}{\partial v_j}\right]_m \to \overline{A}_{ij}\frac{\partial g}{\partial v_j}$ in L^2_{loc}. Passing to a subsequence, if necessary, we have that $\frac{\partial g_m}{\partial v_i}$ is dominated by an L^2 function. Since β'' is bounded, we conclude by dominated convergence that

$$\left[\overline{A}_{ij}\frac{\partial g}{\partial v_j}\right]_m \beta''(g_m)\frac{\partial g_m}{\partial v_i} \to \beta''(g)\overline{A}_{ij}\frac{\partial g}{\partial v_i}\frac{\partial g}{\partial v_j}$$

and

$$\left[\overline{A}_{ij}\frac{\partial g}{\partial v_j}\right]_m \beta'(g_m) \to \overline{A}_{ij}\frac{\partial g}{\partial v_j}\beta'(g) = \overline{A}_{ij}\frac{\partial\beta(g)}{\partial v_j}$$

in L^1_{loc}, thus the above integrals converge to

$$\int_{txv} \beta''(g)\overline{A}_{ij}\frac{\partial g}{\partial v_i}\frac{\partial g}{\partial v_j}\varphi + \int_{txv} \overline{A}_{ij}\frac{\partial\beta(g)}{\partial v_j}\frac{\partial\varphi}{\partial v_i}$$

and similarly, one can prove that

$$\int_{txv} \left[\overline{B}_i g(1-\varepsilon g)\right]_m \left(\beta''(g_m)\frac{\partial g_m}{\partial v_i}\varphi + \beta'(g_m)\frac{\partial\varphi}{\partial v_i}\right) \to \int_{txv} \overline{B}_i g(1-\varepsilon g)\left(\beta''(g)\frac{\partial g}{\partial v_i}\varphi + \beta'(g)\frac{\partial\varphi}{\partial v_i}\right).$$

However, the limit distribution we have above is not yet in the same form as that found in equation (29). Consider then $(g_n)_n$ a sequence of $C_c^\infty((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ functions such that $g_n \to g$ in $L^2_{loc}((0,\infty) \times \mathbb{R}^N_x; H^1_{loc}(\mathbb{R}^N_v))$. For each n, notice we have

$$\beta'(g_n)\frac{\partial}{\partial v_i}\left[\overline{B}_i g_n(1-\varepsilon g_n)\right] = \frac{\partial}{\partial v_i}\left[\overline{B}_i(1-2\varepsilon g_n)\beta(g_n) + 2\varepsilon\overline{B}_i B(g_n)\right] +$$
$$\overline{C}\left[g_n(1-\varepsilon g_n)\beta'(g_n) - (1-2\varepsilon g_n)\beta(g_n) - 2\varepsilon B(g_n)\right],$$

but also

$$\beta'(g_n)\frac{\partial}{\partial v_i}\left[\overline{B}_i g_n(1-\varepsilon g_n)\right] = \frac{\partial}{\partial v_i}\left[\overline{B}_i g_n(1-\varepsilon g_n)\beta'(g_n)\right] - \overline{B}_i g_n(1-\varepsilon g_n)\beta''(g_n)\frac{\partial g_n}{\partial v_i},$$

so passing both sides to the limit in the sense of distributions we conclude that

$$-\int_{txv} \overline{B}_i g(1-\varepsilon g)\left(\beta''(g)\frac{\partial g}{\partial v_i}\varphi + \beta'(g)\frac{\partial\varphi}{\partial v_i}\right) = -\int_{txv} \left[\overline{B}_i(1-2\varepsilon g)\beta(g) + 2\varepsilon\overline{B}_i B(g)\right]\frac{\partial\varphi}{\partial v_i}$$
$$+ \int_{txv} \overline{C}\left[g(1-\varepsilon g)\beta'(g) - (1-2\varepsilon g)\beta(g) - 2\varepsilon B(g)\right]\varphi,$$

for every $\varphi \in C_c^\infty((0,\infty) \times \mathbb{R}_{x,v}^{2N})$ and therefore

$$-\int_{txv} \beta_\delta(g)\frac{\partial\varphi}{\partial t} - \int_{txv} \beta_\delta(g)v \cdot \nabla_x\varphi = \int_{txv} \overline{A}_{ij}\beta_\delta(g)\frac{\partial^2\varphi}{\partial v_i \partial v_j}$$
$$+ \int_{txv} \left[\frac{\partial\overline{A}_{ij}}{\partial v_j}\beta_\delta(g) + \overline{B}_i(1-2\varepsilon g)\beta_\delta(g) + 2\varepsilon\overline{B}_i B(g)\right]\frac{\partial\varphi}{\partial v_i} - \int_{txv} \beta_\delta''(g)\overline{A}_{ij}\frac{\partial g}{\partial v_i}\frac{\partial g}{\partial v_j}\varphi$$
$$+ \int_{txv} \overline{C}\left[g(1-\varepsilon g)\beta_\delta'(g) - (1-2\varepsilon g)\beta_\delta(g) - 2\varepsilon B(g)\right]\varphi$$

for every $\varphi \in C_c^\infty((0,\infty) \times \mathbb{R}_{x,v}^{2N})$.

For the reverse implication, if g satisfies the above equation for every $\delta > 0$, taking $\delta_n = 1/n \to 0$ and noting β_n for β_{δ_n} we have that, almost everywhere,

$$\beta_n(g) \to g,\ \beta_n'(g) \to 1,\ \beta_n''(g) \to 0$$

and from dominated convergence, $B_n(g) \to g^2/2$ a.e. This implies, given that $g \in L^\infty$, that the left-hand side converges to

$$-\int_{txv} g\frac{\partial\varphi}{\partial t} - \int_{txv} gv \cdot \nabla_x\varphi.$$

For the right-hand side we have that,

$$\overline{A}_{ij}\beta_n(g) \to \overline{A}_{ij}g$$
$$\frac{\partial\overline{A}_{ij}}{\partial v_j}\beta_n(g) + \overline{B}_i(1-2\varepsilon g)g + 2\varepsilon\overline{B}_i B(g) \to \frac{\partial\overline{A}_{ij}}{\partial v_j}g + \overline{B}_i(1-\varepsilon g)g$$
$$\beta_n''(g)\overline{A}_{ij}\frac{\partial g}{\partial v_i}\frac{\partial g}{\partial v_j} \to 0$$
$$\overline{C}\left[g(1-\varepsilon g)\beta_n'(g) - (1-2\varepsilon g)\beta_n(g) - 2\varepsilon B(g)\right] \to 0$$

almost everywhere, but since $|\beta_n(g)| \le g$, $|\beta_n'(g)| \le 1$, $|\beta_n''(g)| \le 2$, $|B_n(g)| \le g^2/2$ and $g \in L^\infty((0,\infty) \times \mathbb{R}_{x,v}^{2N})$, these convergences actually hold in L_{loc}^1. □

An immediate corollary of the above result is that although we don't know whether weak LFD solutions are also renormalized solutions, the solutions in an approximation scheme definitely are.

Lemma 2. *Let f be a suitable solution of LFD, with $f_m \to f$ an approximating scheme, as in Definition 3. Then, taking $\beta_\delta(t) = \frac{t}{1+\delta t}$ we have, for each n,*

$$(32)\quad \frac{\partial\beta_\delta(f_m)}{\partial t} + v \cdot \nabla_x\beta_\delta(f_m) = \frac{\partial}{\partial v_i}\left\{\overline{a}_{ij}^m\frac{\partial\beta_\delta(f_m)}{\partial v_j} - \overline{b}_i^m(1-2\varepsilon f_m)\beta_\delta(f_m) - 2\varepsilon\overline{b}_i^m B_\delta(f_m)\right\}$$
$$- \beta_\delta''(f_m)\overline{a}_{ij}^m\frac{\partial f_m}{\partial v_i}\frac{\partial f_m}{\partial v_j} - \overline{c}^m\left[f_m(1-\varepsilon f_m)\beta_\delta'(f_m) - (1-2\varepsilon f_m)\beta_\delta(f_m) - 2\varepsilon B_\delta(f_m)\right]$$

in $\mathcal{D}'([0,T] \times \mathbb{R}_{x,v}^{2N})$, where $\overline{a}^m$ and $\overline{b}^m$ are the same as in equation (16) and $\overline{c}^m = \frac{\partial\overline{b}_i^m}{\partial v_i}$.

5. Diagonal compactness of approximating schemes

The existence results for the Landau equation in [13] and Landau-Fermi-Dirac in [11] are, in essence, compactness results for approximation schemes. For example, for the Landau-Fermi-Dirac equation, it is shown that a sequence of solutions in $L_{loc}^2((0,\infty); L^2(\mathbb{R}_x^N; H^1(\mathbb{R}_v^N)))$ to an approximate equation is strongly compact in $L_{loc}^1((0,\infty); L^1(\mathbb{R}_{x,v}^{2N}))$, which allows us to pass the approximate equation to the limit.

The idea then is to take advantage of this compactness of the approximation schemes to extract the compactness we want for a semi-classical limit. However, since the LFD existence theorem is proved for a fixed quantum parameter, we can't use it directly for a semi-classical limit.

Consider then a sequence of vanishing quantum parameters, $\delta_n \to 0$ and, for each n, let f^n be a suitable solution of LFD, with $f_m^n \to f^n$ being an approximation scheme for this solution. We want to prove that the diagonal of the diagram (21) is strongly compact, that is, given a sequence $m_n \to \infty$, the sequence of approximate solutions $(f_{m_n}^n)_n$ is strongly compact in L^1. Once this diagonal has been established, we can choose f_m^n closer and closer to f^n, and in the end we will have that the f^n converge "in tow", culminating in

Lemma 3. *Let (f^n) be a sequence of suitable weak solutions to (4). For each n, consider an approximating scheme $f_m^n \to f^n$ satisfying (20). If the diagonal sequence $(f_{m_n}^n)_n$ is weakly compact in $L_{loc}^1((0,\infty) \times \mathbb{R}_{x,v}^{2N})$ and*

$$\overline{a}^{n,m_n}, \ \mathrm{div}_v \, \overline{a}^{n,m_n}, \ \overline{b}^{n,m_n}, \ \overline{c}^{n,m_n}$$

are bounded sequences in $L_{loc}^1((0,\infty) \times \mathbb{R}_{x,v}^{2N})$, then $(f_{m_n}^n)_n$ is strongly compact in $L_{loc}^1((0,\infty) \times \mathbb{R}_{x,v}^{2N})$.

The outline of the proof of Lemma 3 is very reminiscent of the proof of the existence of weak solutions to the LFD equation. First, we prove the compactness of velocity averages of the solution sequence, which gives us compactness in the t and x variables. Then we study the parabolic part of the equation in order to get compactness in the v variable and finally a proposition will allow us to join the two results and get full compactness in L^1 for these solutions.

5.1. **Velocity Averaging.** In the study of transport equations, the averaging lemmas constitute a whole body of results stating roughly that, if f is a solution to a transport equation $\partial_t f + v \cdot \nabla f = g$ such that f and g are bounded in L^2, for example, then the velocity average

$$\int f \varphi \, dv$$

should be bounded in a more regular space (in the above example, it is bounded in $H^{1/2}$). Hence for a bounded sequence of solutions f^n then, we expect the velocity averages to be compact in L^2.

First shown independently by Agoshkov [2] and Golse, Perthame and Sentis in [6] in an L^2 setting, averaging lemmas are now a standard technique in the study of kinetic equations. The version we'll use here is a simple corollary of Theorem 1.1.8 of [3], which reads

Theorem 3. *Let Ω be an open set of $\mathbb{R}_t \times \mathbb{R}_x^N$, $(f_n)_n$ a sequence of functions bounded in $L_{loc}^p(\Omega \times \mathbb{R}^M)$, with $p > 1$, verifying*

$$\partial_t f_n + v \cdot \nabla_x f_n = \sum_{|\alpha| \le m} \partial_v^\alpha g_n^{(\alpha)} \ in \ \Omega \times \mathbb{R}^M$$

for some $m \in \mathbb{N}$, where $(g_n^{(\alpha)})_n$ is a bounded sequence in the space of measures $\mathcal{M}_{loc}(\Omega \times \mathbb{R}^M)$ for each $|\alpha| \le m$. For any $\psi \in C_c^\infty(\mathbb{R}^M)$, consider the velocity averages

$$\rho_\psi^n(t,x) = \int f_n(t,x,v) \psi(v) \, dv.$$

Then, the sequence $(\rho_\psi^n)_n$ is compact in $L_{loc}^q(\Omega)$ for any $q < p$.

In order to apply this theorem to the renormalized equation (32) we have deduced in the last section, we first show an estimate for the quadratic term in the derivatives on the right-hand side of this equation.

Lemma 4. *Let (f^n) be a sequence of suitable weak solutions to (4). For each n, consider an approximating scheme $f_m^n \to f^n$, as in Definition 3. If the sequences*

$$\overline{a}^{n,m_n}, \ \mathrm{div}_v \, \overline{a}^{n,m_n}, \ \overline{b}^{n,m_n}, \ \overline{c}^{n,m_n}$$

are uniformly bounded in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ then for every $T, R > 0$,

$$\iiint_{(0,T) \times B_R \times B_R} \overline{a}_{ij}^{n,m_n} \frac{\partial \beta(f_{m_n}^n)}{\partial v_i} \frac{\partial \beta(f_{m_n}^n)}{\partial v_j} \, dt dx dv \leq C_{T,R}$$

where $C_{T,R} > 0$ is a universal constant, which doesn't depend on n and $\beta(t) = \frac{t}{1+t}$.

Proof. In equation (32), choose $\beta(t) = \frac{t}{1+t}$ as non-linearity and $\varphi \geq 0$ such that $\varphi \equiv 1$ in $(0,T) \times B_R \times B_R$ as a test function. This way, we have, since $\beta''(t) = -\frac{2}{(1+t)^3}$

$$2 \int_{txv} \overline{a}_{ij}^{n,m} \frac{1}{(1+f_m^n)^3} \frac{\partial f_m^n}{\partial v_i} \frac{\partial f_m^n}{\partial v_j} \varphi \leq \left| \int_{xv} \beta(f_m^n)(0)\varphi(0) \right| + \left| \int_{txv} \beta(f_m^n) \frac{\partial \varphi}{\partial t} \right|$$

$$+ \left| \int_{txv} \beta(f_m^n) v \cdot \nabla_x \varphi \right| + \left| \int_{txv} \overline{a}_{ij}^{n,m} \beta(f_m^n) \frac{\partial^2 \varphi}{\partial v_i \partial v_j} \right|$$

$$+ \left| \int_{txv} \left[\frac{\partial \overline{a}_{ij}^{n,m}}{\partial v_j} \beta(f_m^n) + \overline{b}_i^{m,n}(1 - 2\varepsilon_n f_m^n)\beta(f_m^n) + 2\varepsilon_n \overline{b}^{m,n} B(f_m^n) \right] \frac{\partial \varphi}{\partial v_i} \right|$$

$$+ \left| \int_{txv} \overline{c}^{n,m} \left[f_m^n(1 - \varepsilon_n f_m^n)\beta'(f_m^n) - (1 - 2\varepsilon_n f_m^n)\beta(f_m^n) - 2\varepsilon_n B(f_m^n) \right] \varphi \right|.$$

Since $|\beta(t)| \leq 1$, the first four terms in the right-hand side are bounded by

$$\|\varphi(0)\|_{L^1_{xv}} + \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^1_{txv}} + \|v \cdot \nabla_x \varphi\|_{L^1_{txv}} + \left\| \overline{a}_{ij}^{n,m} \right\|_{L^1(\mathrm{supp}\,\varphi)} \left\| \frac{\partial^2 \varphi}{\partial v_i \partial v_j} \right\|_{L^\infty}.$$

Now, since $0 \leq f_m^n \leq \varepsilon_n^{-1}$ we have $|1 - 2\varepsilon_n f_m^n| \leq 1$ and for this β,

$$|\varepsilon_n B(f_m^n)| = \varepsilon_n(f_m^n - \log(1 + f_m^n)) \leq \varepsilon_n f_m^n \leq 1$$

hence, the fifth term is bounded by

$$\left(\|\mathrm{div}_v \, \overline{a}^{n,m}\|_{L^1(\mathrm{supp}\,\varphi)} + 2\|\overline{b}_i^{n,m}\|_{L^1(\mathrm{supp}\,\varphi)} \right) \|\varphi\|_{L^\infty}.$$

Finally, using that $0 \leq t\beta'(t) = \frac{t}{(1+t)^2} \leq 1$ for $t \geq 0$ and $|1 - \varepsilon_n f_m^n| \leq 1$, we bound the last term by

$$4\|\overline{c}^{n,m}\|_{L^1(\mathrm{supp}\,\varphi)}\|\varphi\|_{L^\infty}.$$

The result then follows since $(\beta'(t))^2 = \frac{1}{(1+t)^4} \leq \frac{1}{(1+t)^3}$. $\qquad\square$

We could use Lemma 4 directly, together with the estimates we have for the f^n solutions, to deduce the compactness of the velocity averages of $\beta(f^n)$. However, as in the proof for the existence of global solutions to the LFD equation, it turns out that the quantity we must prove compactness for the velocity averages is $f^n(1 - \varepsilon_n f^n)$. We therefore need to do some extra work to get the compactness from one to the other.

Lemma 5. *Let $\varepsilon_n \to 0$ be a vanishing sequence of quantum parameters and for each n let f^n be a suitable solution of LFD with quantum parameter ε_n, with $f_m^n \to f^n$ an approximating scheme, as in Definition 3. Define $F_m^n \equiv f_m^n(1 - \varepsilon_n f_m^n)$.*

Let $m_n \to \infty$ be such that $(f_{m_n}^n)_n$ is weakly compact in $L_{loc}^1((0,\infty); L^1(\mathbb{R}_{x,v}^{2N}))$, then the velocity averages

$$\int F_{m_n}^n \varphi \, dv \quad and \quad \int \beta_\delta(f_{m_n}^n) \varphi \, dv$$

are compact in $L_{loc}^1((0,\infty) \times \mathbb{R}_x^N)$, for every $\varphi \in C_c^\infty(\mathbb{R}_v^N)$.

Proof. Throughout this proof, we will notate the sequence $f_{m_n}^n$ as f^n, as well as the coefficients $\overline{a}_m^n$, $\overline{b}_m^n$ and $\overline{c}_m^n$ as $\overline{a}^n$, $\overline{b}^n$ and $\overline{c}^n$, respectively.

This proof is divided into three parts. First, we will show the compactness of the velocity averages of $\beta_\delta(f^n)$, using the renormalized formulation. We then proceed by showing that the velocity averages of $\beta_\delta(f^n)$ converge to the same limit as the velocity averages of $\beta_\delta(F^n)$ and finally we relax the value of $\delta > 0$ to achieve compactness for the velocity averages of F^n.

Part I: Velocity averages of $\beta_\delta(f^n)$

Equation (32) can be rewritten as

$$\frac{\partial \beta_\delta(f^n)}{\partial t} + v \cdot \nabla_x \beta_\delta(f^n) = \frac{\partial^2 G_{ij}^n}{\partial v_i \partial v_j} - \frac{\partial G_i^n}{\partial v_i} + G^n$$

where the functions in the right-hand side are given by

$$G_{ij}^n = \overline{a}_{ij}^n \beta_\delta(f^n)$$

$$G_i^n = \frac{\partial \overline{a}_{ij}^n}{\partial v_j} \beta_\delta(f^n) + \overline{b}_i^n(1 - 2\varepsilon_n f^n)\beta_\delta(f^n) + 2\varepsilon_n \overline{b}_i^n B_\delta(f^n)$$

$$G^n = -\overline{c}^n \left[f^n(1 - \varepsilon_n f^n)\beta_\delta'(f^n) - (1 - 2\varepsilon_n f^n)\beta_\delta(f^n) - 2\varepsilon_n B_\delta(f^n) \right] - \beta_\delta''(f^n)\overline{a}_{ij}^n \frac{\partial f^n}{\partial v_i} \frac{\partial f^n}{\partial v_j}$$

Now, since $\beta_\delta(t) \leq \delta^{-1}$, $|1 - 2\varepsilon_n f^n| \leq 1$ and

$$|\varepsilon_n B_\delta(f^n)| = \varepsilon_n \left(\frac{f^n}{\delta} - \frac{\log(\delta f^n + 1)}{\delta^2} \right) \leq \varepsilon_n \delta^{-1} f^n \leq \delta^{-1},$$

it follows that the first two functions satisfy

$$|G_{ij}^n| \leq \delta^{-1}|\overline{a}_{ij}^n| \quad and \quad |G_i^n| \leq \delta^{-1}\left| \frac{\partial \overline{a}_{ij}^n}{\partial v_j} \right| + 3\delta^{-1}|\overline{b}_i^n|$$

and thus these sequences are bounded in $L_{loc}^1((0,\infty) \times \mathbb{R}_x^N)$. Moreover, since $t\beta_\delta'(t) = \frac{t}{(1+\delta t)^2} \leq \frac{1}{\delta}$, it follows that

$$|G^n| \leq \frac{4}{\delta}|\overline{c}^n| - \beta_\delta''(f^n)\overline{a}_{ij}^n \frac{\partial f^n}{\partial v_i} \frac{\partial f^n}{\partial v_j}$$

and from Lemma 4, this sequence is bounded in $L_{loc}^1((0,\infty) \times \mathbb{R}_{x,v}^{2N})$. Since $(\beta_\delta(f^n))_n$ is bounded in $L^\infty((0,\infty) \times \mathbb{R}_{x,v}^{2N})$, Theorem 3 implies that the velocity averages $\int \beta_\delta(f^n)\varphi \, dv$ are compact in $L_{loc}^1((0,\infty) \times \mathbb{R}_{x,v}^{2N})$. We then pass to a convergent subsequence.

Part II: Velocity averages of $\beta(F^n)$

The next part of the proof consists of passing from the compactness of the $\beta(f^n)$ averages to the compactness of the $\beta(F^n)$ averages. Notice that

$$|\beta_\delta(F^n) - \beta_\delta(f^n)| = \varepsilon_n f^n \frac{f^n}{1 + \delta f^n} \frac{1}{1 + \delta F^n} \leq \delta^{-1}\varepsilon_n f^n$$

and thus we may estimate the difference of norms as

$$\int_{tx} \left| \int_v \beta(F^n)\varphi - \int_v \beta(f^n)\varphi \right| \leq \int_{txv} \delta^{-1}\varepsilon_n f^n|\varphi| \leq \delta^{-1}\varepsilon_n \|\varphi\|_{L_v^\infty} \|f^n\|_{L_{txv}^1}$$

and since $\|f^n\|_{L^1_{txv}} \le C$, the above quantity converges to zero as $n \to \infty$. Since the velocity averages of $\beta_\delta(f^n)$ are compact, the above argument not only shows that the velocity averages of $\beta_\delta(F^n)$ are also compact, but that the two limits coincide.

Part III: Velocity averages of F^n

In this last step, we will use the fact that the $\delta > 0$ taken so far is arbitrary to obtain the compactness of the averages of F^n. We have, for any $K > 1$,

$$|\beta_\delta(F^n) - F^n| = \frac{\delta(F^n)^2}{1 + \delta F^n} = \frac{\delta(F^n)^2}{1 + \delta F^n}\mathbb{1}_{F^n \le K} + \frac{\delta(F^n)^2}{1 + \delta F^n}\mathbb{1}_{F^n > K}.$$

Using that $t \mapsto \frac{t}{1+\delta t}$ is an increasing function bounded by $1/\delta$ and that $F^n = f^n(1 - \varepsilon_n f^n) \le f^n$, it follows that

$$|\beta_\delta(F^n) - F^n| \le \frac{K}{1 + \delta K}\delta F^n \mathbb{1}_{F^n \le K} + F^n \mathbb{1}_{F^n > K}$$
$$\le K\delta f^n + f^n \mathbb{1}_{f^n > K}$$

and therefore

$$\int_{tx}\left|\int_v \beta_\delta(F^n)\varphi - \int_v F^n\varphi\right| \le K\delta\|\varphi\|_{L^\infty_v}\int_{txv}f^n + \|\varphi\|_{L^\infty_v}\int_{txv}f^n \mathbb{1}_{f^n > K}.$$

But since $(f^n)_n$ is uniformly integrable, for every $\varepsilon > 0$ there exists some $K_\varepsilon > 0$ such that

$$\sup_n \int_{txv} f^n \mathbb{1}_{f^n > K_\varepsilon} \le \varepsilon.$$

This way, there exists a $C = C(T, \varphi) > 0$ such that for every $\delta \le \varepsilon/K_\varepsilon$,

$$\left\|\int_v \beta_\delta(F^n)\varphi - \int_v F^n\varphi\right\|_{L^1} \le C(T, \varphi)\varepsilon, \quad \forall n \in \mathbb{N}.$$

For each $m \in \mathbb{N}$, let $\delta_m > 0$ be such that

$$\left\|\int_v \beta_{\delta_m}(F^n)\varphi - \int_v F^n\varphi\right\|_{L^1} \le \frac{1}{m},$$

for every $n \in \mathbb{N}$. Since the velocity averages for $\beta_\delta(f^n)$ are compact we conclude that, for each m, the average $\int_v \beta_{\delta_m}(F^n)\varphi$ will converge, up to a subsequence. Let us notate $\rho^n_m = \int_v \beta_{\delta_m}(F^n)\varphi$. We will use a diagonal argument to construct a single subsequence that converges for *every* m.

We have $(\rho^n_1)_n$ compact, thus there exists some sequence $k_1(n)$ such that $(\rho^{k_1(n)}_1)_n$ converges. Next, since $(\rho^{k_1(n)}_2)_n$ compact, we take a k_2 subsequence of k_1 such that $(\rho^{k_2(n)}_2)_n$ converges and so on.

This way, we construct nested sequences k_m such that $(\rho^{k_m(n)}_\ell)_n$ converges, for every $\ell \le m$. Consider the diagonal sequence $(k_n(n))_n$. By definition, from index ℓ on, this is a subsequence of $(k_\ell(n))_n$, and thus $(\rho^{k_n(n)}_\ell)_n$ converges for every $\ell \in \mathbb{N}$.

We then pass to this subsequence, notating simply F^n instead of $F^{\varphi_n(n)}$. For every $n, m \in \mathbb{N}$, we have

$$\left\|\int_v F^n\varphi - \int_v F^m\varphi\right\|_{L^1} \le \left\|\int_v F^n\varphi - \int_v \beta_{\delta_k}(F^n)\varphi\right\|_{L^1} + \left\|\int_v \beta_{\delta_k}(F^n)\varphi - \int_v \beta_{\delta_k}(F^m)\varphi\right\|_{L^1}$$
$$+ \left\|\int_v \beta_{\delta_k}(F^m)\varphi - \int_v F^m\varphi\right\|_{L^1}$$
$$\le \frac{2}{k} + \left\|\int_v \beta_{\delta_k}(F^n)\varphi - \int_v \beta_{\delta_k}(F^m)\varphi\right\|_{L^1}$$

and since $(\int \beta_{\delta_k}(F^n)\varphi \, dv)_n$ is a Cauchy sequence in L^1, we have that $(\int F^n\varphi \, dv)_n$ is also a Cauchy sequence, and thus converges. $\square$

5.2. **Almost everywhere compactness.** Once we have found some compactness in the variables t and x through velocity averaging, the aim now is to find compactness in the variable v.

The traditional way of doing this, widely used in the study of parabolic equations, is to search for an ellipticity estimate for the diffusion matrices $\bar{a}_{ij}^n$ that is uniform in n, which together with Lemma 4 would give us a bound on the derivatives of f. Such an estimate. however, would fall apart if we had, for example, $\int_v f^n(1 - \varepsilon_n f^n) = 0$ at some (t,x), as this implies $\bar{a}^n(t,x,v) = 0$ at that point. Thus, this estimate depends primarily on the fact that we don't have a quantum vacuum at any (t,x), which seems to us to be a very difficult estimate to achieve.

But it turns out that such an elliptic estimate is not necessary, and we obtain sufficient compactness in v using only a partial notion of ellipticity satisfied by $\bar{a}$, namely Lemma 2 of [11], which is rewritten, for our case, as

Lemma 6. *Fix $T, R > 0$. Define $F_m^n \equiv f_m^n(1 - \varepsilon_n f_m^n)$. Suppose there exists some $F \in L^\infty((0,\infty) \times \mathbb{R}_{x,v}^{2N}) \cap L_{loc}^1((0,\infty); L^1(\mathbb{R}_{x,v}^{2N}))$ such that*

$$\int F_{m_n}^n \varphi \, dv \to \int F\varphi \, dv \ in \ L^1([0,T] \times B_R)$$

for every $\varphi \in L^1(\mathbb{R}_v^N)$. Let $\bar{a}^{n,m}$ be a sequence of matrices satisfying (20) and

$$K_\alpha \equiv \left\{ (t,x) \in [0,T] \times B_R : \int F \, dv > \alpha \right\}.$$

Then, for every $\alpha, \varepsilon > 0$ there exists a measurable set $|E| < \varepsilon$ such that

$$\bar{a}_{ij}^{n,m_n}(t,x,v)\eta_i\eta_j \geq C(\alpha,\varepsilon)|\eta|^2,$$

for $(x,t,v,\eta) \in (K_\alpha \cap E^c) \times B_R \times \mathbb{R}^N$ and $n \geq n_0(\varepsilon,\alpha)$.

This ellipticity, together with Lemma 4, gives us an estimate for the derivatives of f in v, which incurs a compactness in the variable v. Having already established compactness in t and x from the averaging lemmas in the last section, the next question is whether we can unify these two results into a single compactness in the three variables t, x, v.

The next proposition, which was also stated and proven in [11], shows us that this question is answered in the affirmative.

Proposition 2. *Let $T > 0$ and $(\Phi^n)_n$ be a sequence of functions such that $\Phi^n \overset{*}{\rightharpoonup} \Phi$ in $L^\infty((0,\infty) \times \mathbb{R}_{x,v}^{2N})$. Suppose that, for each $T, R > 0$,*

    *i. **Quasi-bounded in v:** for every $\varepsilon, \alpha > 0$, there exists a $C = C(\varepsilon,\alpha)$ and a measurable set E, with $|E| \leq \varepsilon$ such that*

$$\iiint_{(K_\alpha \cap E^c) \times B_R} |\nabla_v \Phi^n|^2 \, dtdxdv \leq C(\varepsilon,\alpha)$$

    where $K_\alpha = \left\{ (t,x) \in [0,T] \times B_R : \int \Phi \, dv > \alpha \right\}$,

    *ii. **Velocity averages:** for every $\varphi \in C_c^\infty(\mathbb{R}_v^N)$, $(\int \Phi^n\varphi \, dv)_n$ is compact in $L^1([0,T] \times B_R)$.*

Thus, passing to a subsequence, we have $\Phi^n \to \Phi$ almost everywhere in $(0,\infty) \times \mathbb{R}_{x,v}^{2N}$.

Now we can finally gather all the above results to prove the proposition of this section

Proof of Lemma 3. First, it follows from Lemma 4 that for every $T, R > 0$, there exists a $C_{T,R} > 0$ such that

$$\iiint_{(0,T) \times B_R \times B_R} \bar{a}_{ij}^{n,m_n} \frac{\partial \beta_1(f_{m_n}^n)}{\partial v_i} \frac{\partial \beta_1(f_{m_n}^n)}{\partial v_j} \, dt dx dv \leq C_{T,R}.$$

One would then like to chain together Lemmas 5 and 6. Since we have (**??**), for every diagonal sequence m_n such that $m_n \geq M_{n,T}$ we have a subsequence such that $f_{m_n}^n \rightharpoonup f$ weakly in $L_{loc}^1((0,\infty) \times \mathbb{R}_{x,v}^{2N})$. Passing to a further subsequence in n, we have that $\beta_\delta(F_{m_n}^n) \overset{*}{\rightharpoonup} b$ weakly in $L^\infty((0,\infty) \times \mathbb{R}_{x,v}^{2N})$, for some bounded function b.

Note that we cannot apply directly Proposition 2 here, since, the K_α that is given by Lemma 6, defined in terms of $\int F \, dv$, is not necessarily the same as the one in the statement of Proposition 2, which is written in terms of $\int b \, dv$.

To fix this, notice that

$$|\beta_\delta(F_{m_n}^n) - \beta_\delta(f_{m_n}^n)| \leq \delta^{-1} \varepsilon_n f_{m_n}^n,$$

which implies

$$\left| \int \beta_\delta(F_{m_n}^n) \varphi - \int \beta_\delta(f_{m_n}^n) \varphi \right| \leq \delta^{-1} \varepsilon_n \int f_{m_n}^n \varphi$$

for every $\varphi \in C^\infty((0,\infty) \times \mathbb{R}_{x,v}^{2N})$, and thus this quantity converges to zero as $n \to \infty$. Using that $\beta_\delta(t) \leq t$ it follows that, for every $\varphi \in C^\infty$ such that $\varphi \geq 0$ satisfies

$$\int_{txv} \beta_\delta(F_{m_n}^n) \varphi \leq \int_{txv} F_{m_n}^n \varphi,$$

which passing to the limit implies that $b \leq F$ almost everywhere.

Thus, we have that

$$K_\alpha' \equiv \left\{ (t,x) \in [0,T] \times B_R : \int b \, dv > \alpha \right\} \subset \left\{ (t,x) \in [0,T] \times B_R : \int F \, dv > \alpha \right\} \equiv K_\alpha$$

and therefore Lemma 6 will give us that, in particular,

$$\iiint_{(K_\alpha' \cap E^c) \times B_R} \left| \nabla_v \beta_1(f_{m_n}^n) \right|^2 \leq C(\varepsilon, \alpha),$$

and then since the velocity averages $\int \beta_1(f_{m_n}^n) \varphi \, dv$ are compact, Proposition 2 then implies that, passing to a subsequence, we have $\beta_1(f_{m_n}^n)$ converging almost everywhere in $(0,\infty) \times \mathbb{R}_{x,v}^{2N}$, which implies that $f_{m_n}^n$ also converges a.e. in this space.

Since this sequence is weakly compact in $L_{loc}^1((0,\infty); L^1(\mathbb{R}_{x,v}^{2N}))$, we can then infer the strongly compactness in this space from Scheffé's lemma. $\qquad\square$

6. Proof of Theorem 1

We now have all the necessary ingredients for the proof of Theorem 1, which we will divide into two sections. In the first part, we'll apply the compactness results we've seen so far to show that the solutions of the Landau-Fermi-Dirac equation converge to renormalized solutions of the Landau equation, as in Villani's definition. In the second part of the proof, we will show the convergence of the conservation laws and the dissipation of entropy, showing that the entropy inequality is still valid for solutions in the semi-classical limit.

6.1. **Convergence to renormalized solutions.** For each n, let $(f^n_m)_m$ be an approximating scheme converging to f^n, with coefficients $\overline{a}^{n,m}$ and $\overline{b}^{n,m}$.

Let $B_n = (0,n) \times B_x(0,n) \times B_v(0,n)$ be the t,x,v ball with radius n. Since $f^n_m \to f^n$ in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ we can suppose, up to passing to subsequences in m, that

$$\|f^n - f^n_m\|_{L^1} \le \frac{1}{n} \tag{33}$$

for every $m \in \mathbb{N}$, as well as

$$\|\overline{a}^{n,m}_{ij} - \overline{a}^n_{ij}\|_{L^1(B_n)}, \left\|\frac{\partial \overline{a}^{n,m}_{ij}}{\partial v_j} - \frac{\partial \overline{a}^n_{ij}}{\partial v_j}\right\|_{L^1(B_n)}, \|\overline{b}^{n,m}_i - \overline{b}^n_i\|_{L^1(B_n)}, \|\overline{c}^{n,m} - \overline{c}^n\|_{L^1(B_n)} \le \frac{1}{n}, \tag{34}$$

from the convergences in Definition 3.

This way, by chaining Lemmas 1 and 3 we obtain a diagonal sequence $(f^n_{m_n})_n$ that is strongly compact in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ and we can then pass to a convergent subsequence.

For conciseness, we will notate f^n_n instead of $f^n_{m_n}$, as well as $\overline{a}^{n,n}$ instead of $\overline{a}^{n,m_n}$ for the other coefficients and so on. Passing to a further subsequence we have that

$$f^n_n \to f \text{ in } L^1_{loc}((0,\infty); L^1(\mathbb{R}^{2N}_{x,v})) \text{ and a.e. in } (0,\infty) \times \mathbb{R}^{2N}_{x,v}. \tag{35}$$

Since $f^n_n \in L^2_{loc}((0,T) \times \mathbb{R}^N_x; H^1_{loc}(\mathbb{R}^N_v))$, Proposition 1 ensures that f^n_n satisfies the renormalized formulation, that is, for every $\delta > 0$, taking $\beta_\delta(t) = \frac{t}{1+\delta t}$, we have

$$\frac{\partial \beta_\delta(f^n_n)}{\partial t} + v \cdot \nabla_x \beta_\delta(f^n_n) = \frac{\partial}{\partial v_i} \left\{ \overline{a}^{n,n}_{ij} \frac{\partial \beta_\delta(f^n_n)}{\partial v_j} - \overline{b}^{n.n}_i (1 - 2\varepsilon_n f^n_n) \beta_\delta(f^n_n) - 2\varepsilon_n \overline{b}^{n,n}_i B(f^n_n) \right\}$$
$$- \beta''_\delta(f^n_n) \overline{a}^{n,n}_{ij} \frac{\partial f^n_n}{\partial v_i} \frac{\partial f^n_n}{\partial v_j} - \overline{c}^{n,n} [f^n_n (1 - \varepsilon_n f^n_n) \beta'_\delta(f^n_n) - (1 - 2\varepsilon_n f^n_n) \beta_\delta(f^n_n) - 2\varepsilon_n B_\delta(f^n_n)]$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{2N}_{x,v})$.

From (35) we have $\beta_\delta(f^n_n) \to \beta_\delta(f)$ a.e., thus since $|\beta_\delta(f^n_n)| \le \frac{1}{\delta}$, it follows that

$$\frac{\partial \beta_\delta(f^n_n)}{\partial t} + v \cdot \nabla_x \beta_\delta(f^n_n) \to \frac{\partial \beta_\delta(f)}{\partial t} + v \cdot \nabla_x \beta_\delta(f).$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{2N}_{x,v})$.

Next, let us prove the convergence of the coefficients $\overline{a}^{n,n}_{ij}$, $\frac{\partial \overline{a}^{n,n}_{ij}}{\partial v_j}$, $\overline{b}^{n,n}_i$ and $\overline{c}^{n,n}$.

Consider $\overline{a}^n_{ij} = a_{ij} *_v (f^n(1 - \varepsilon f^n))$. Recall that the convolution of a function g with a signed measure μ is the function defined by

$$(\mu * g)(x) \equiv \int g(x - x_*) d\mu(x_*)$$

and we have the inequality

$$\|\mu * g\|_{L^1} \le \|g\|_{L^1} |\mu|(\mathbb{R}^N).$$

Therefore if $g^n \to g$ in L^1, then $\mu * g^n \to \mu * g$ in L^1. For $\overline{a}^n_{ij}$ we need a strong convergence result for $f^n(1 - \varepsilon_n f^n)$. We have that $f^n(1 - \varepsilon_n f^n) \to f$ almost everywhere. Since for every n we have $|f^n(1 - \varepsilon_n f^n)| \le f^n$, the sequence f^n converges to f almost everywhere and in L^1, thus by dominated convergence, $f^n(1 - \varepsilon_n f^n) \to f$ in L^1.

This implies that $\overline{a}^n_{ij} \to \overline{a}_{ij}$, $\frac{\partial \overline{a}^n_{ij}}{\partial v_j} \to \frac{\partial \overline{a}_{ij}}{\partial v_j}$, $\overline{b}^n_i \to \overline{b}_i$ and $\overline{c}^n \to \overline{c}$ in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$, then using (34), we have that

$$\overline{a}^{n,n}_{ij} \to \overline{a}_{ij}, \frac{\partial \overline{a}^{n,n}_{ij}}{\partial v_j} \to \frac{\partial \overline{a}_{ij}}{\partial v_j}, \overline{b}^{n,n}_i \to \overline{b}_i, \overline{c}^{n,n} \to \overline{c}$$

in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ and, by passing to a further subsequence, a.e. in $(0,\infty) \times \mathbb{R}^{2N}_{x,v}$. Notice that since $|\beta_\delta(f^n_n)| \leq \frac{1}{\delta}$ we have, from dominated convergence, that $\beta_\delta(f_n) \overset{*}{\rightharpoonup} \beta_\delta(f)$ in $L^\infty((0,\infty) \times \mathbb{R}^{2N}_{x,v})$.

Following Remark 1, we consider

$$\overline{a}^{n,n}_{ij} \frac{\partial \beta_\delta(f^n_n)}{\partial v_j} = \frac{\partial}{\partial v_j} \left[\overline{a}^{n,n}_{ij} \beta_\delta(f^n_n) \right] - \frac{\partial \overline{a}^{n,n}_{ij}}{\partial v_j} \beta_\delta(f^n_n),$$

and by weak-strong convergence, $\overline{a}^{n,n}_{ij} \beta_\delta(f^n_n) \to \overline{a}_{ij} \beta_\delta(f)$, $\frac{\partial \overline{a}^{n,n}_{ij}}{\partial v_j} \beta_\delta(f^n_n) \to \frac{\partial \overline{a}_{ij}}{\partial v_j} \beta_\delta(f)$ in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ which implies

$$\overline{a}^{n,n}_{ij} \frac{\partial \beta_\delta(f^n_n)}{\partial v_j} \to \frac{\partial}{\partial v_i} \left[\overline{a}_{ij} \beta_\delta(f) \right] - \frac{\partial \overline{a}_{ij}}{\partial v_j} \beta_\delta(f)$$

which we will denote $\overline{a}_{ij} \frac{\partial \beta_\delta(f)}{\partial v_j}$.

Next, since $0 \leq f^n_n \leq \varepsilon_n^{-1}$, we have that $|1 - 2\varepsilon_n f^n_n| \leq 3$ and dominated convergence implies

$$(1 - 2\varepsilon_n f^n_n) \beta_\delta(f^n_n) \overset{*}{\rightharpoonup} \beta_\delta(f) \text{ in } L^\infty((0,\infty) \times \mathbb{R}^{2N}_{x,v}).$$

Next, there exists a constant $C > 0$ such that $|B_\delta(t)| \leq t/\delta + C$, for every $t \geq 0$, which implies that

$$\varepsilon_n B_\delta(f^n_n) \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,\infty) \times \mathbb{R}^{2N}_{x,v}),$$

again by dominated convergence.

Hence, as before, we have by weak-strong convergence that

$$\frac{\partial}{\partial v_i} \left[\overline{b}^{n,n}_i (1 - 2\varepsilon_n f^n_n) \beta_\delta(f^n_n) + 2\varepsilon_n \overline{b}^{n,n}_i B_\delta(f^n_n) \right] \to \frac{\partial}{\partial v_i} \left[\overline{b}_i \beta_\delta(f) \right]$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{2N}_{x,v})$.

Also, since $|f^n_n(1 - \varepsilon_n f^n_n)| \leq f^n_n$ and $t\beta'_\delta(t) = \frac{t}{(1+\delta t)^2} \leq \frac{1}{4\delta}$ for every $t \geq 0$ we have that

$$f^n_n (1 - \varepsilon_n f^n_n) \beta'_\delta(f^n_n) \overset{*}{\rightharpoonup} f\beta'_\delta(f) \text{ in } L^\infty((0,\infty) \times \mathbb{R}^{2N}_{x,v}).$$

Thus, from weak-strong convergence,

$$\overline{c}^{n,n} \left[f^n_n (1 - \varepsilon_n f^n_n) \beta'_\delta(f^n_n) - (1 - 2\varepsilon_n f^n_n) \beta_\delta(f^n_n) - 2\varepsilon_n B_\delta(f^n_n) \right] \to \overline{c} \left[f\beta'_\delta(f) - \beta_\delta(f) \right]$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{2N}_{x,v})$.

It remains just to prove the convergence of the quadratic term

$$(36) \qquad\qquad -\beta''_\delta(f^n_n) \overline{a}^{n,n}_{ij} \frac{\partial f^n_n}{\partial v_i} \frac{\partial f^n_n}{\partial v_j}$$

in the sense of distributions. Let then

$$\gamma_\delta(t) = -\frac{2\sqrt{2}}{\sqrt{\delta(1 + \delta t)}},$$

notice this implies $(\gamma'_\delta(t))^2 = \frac{2\delta}{(1+\delta t)^3} = -\beta''_\delta(t)$. We can then rewrite

$$\int_{txv} -\beta''_\delta(f^n_n) \overline{a}^{n,n}_{ij} \frac{\partial f^n_n}{\partial v_i} \frac{\partial f^n_n}{\partial v_j} \varphi = \int_{txv} \overline{a}^{n,n}_{ij} \frac{\partial \gamma_\delta(f^n_n)}{\partial v_i} \frac{\partial \gamma_\delta(f^n_n)}{\partial v_j} \varphi,$$

for every $\varphi \in C_c^\infty((0,\infty) \times \mathbb{R}_{x,v}^{2N})$ such that $\varphi \geq 0$ and by (18) this is larger than

$$\int_{txv} [a_{m_n} *_v (f_n^n(1 - \varepsilon_n f_n^n))] \frac{\partial \gamma_\delta(f_n^n)}{\partial v_i} \frac{\partial \gamma_\delta(f_n^n)}{\partial v_j} \varphi$$

$$= \int_{txv} \int_{v_*} a_{m_n}(v - v_*) f_n^n(v_*)(1 - \varepsilon_n f_n^n(v_*)) \frac{\partial \gamma_\delta(f_n^n)}{\partial v_i} \frac{\partial \gamma_\delta(f_n^n)}{\partial v_j} \varphi$$

$$= \int_{txv} \int_{v_*} \left| \sqrt{a_{m_n}(v - v_*)} \sqrt{f_n^n(v_*)(1 - \varepsilon_n f_n^n(v_*))} \nabla_v \gamma_\delta(f_n^n) \right|^2 \varphi,$$

where $\sqrt{a_{m_n}(v - v_*)}$ denotes the matrix square root of $a_{m_n}(v - v_*)$. For conciseness of notation, let us write a_n instead of a_{m_n}. Hence, there exists a positive Radon measure

$$(37) \qquad \nu^n \equiv -\beta_\delta''(f_n^n) \bar{a}_{ij}^{n,n} \frac{\partial f_n^n}{\partial v_i} \frac{\partial f_n^n}{\partial v_j} - \left| \sqrt{a_n(v - v_*)} \sqrt{f_n^n(v_*)(1 - \varepsilon_n f_n^n(v_*))} \nabla_v \gamma_\delta(f_n^n) \right|^2$$

and we have

$$\langle \nu^n, \varphi \rangle \leq \int_{txv} -\beta_\delta''(f_n^n) \bar{a}_{ij}^{n,n} \frac{\partial f_n^n}{\partial v_i} \frac{\partial f_n^n}{\partial v_j} \varphi.$$

This last quantity is uniformly bounded in n thanks to Lemma 4, thus up to passing to a subsequence in n, there exists a positive measure μ such that

$$\nu^n \rightharpoonup \nu \text{ in } \mathcal{M}_{loc}((0,\infty) \times \mathbb{R}_{xvv_*}^{3N})$$

where $\mathbb{R}_{xvv_*}^{3N}$ denotes the space $\mathbb{R}_x^N \times \mathbb{R}_v^N \times \mathbb{R}_{v_*}^N$.

Then, we rewrite the term inside the square of (37) as

$$(38) \quad \sqrt{a_n(v - v_*)} \sqrt{f_n^n(v_*)(1 - \varepsilon_n f_n^n(v_*))} \nabla_v \gamma_\delta(f_n^n) =$$

$$\text{div}_v \left[\sqrt{a_n(v - v_*)} \sqrt{f_n^n(v_*)(1 - \varepsilon_n f_n^n(v_*))} \gamma_\delta(f_n^n) \right]$$

$$- \text{div}_v \left[\sqrt{a_n(v - v_*)} \right] \sqrt{f_n^n(v_*)(1 - \varepsilon_n f_n^n(v_*))} \gamma_\delta(f_n^n).$$

The matrices a_n have a fixed form, given by (18) and therefore by the uniqueness of the square root of positive matrices,

$$(39) \qquad \sqrt{a_n(z)} = \sqrt{\Gamma_n(|z|)} \left(I - \frac{z \otimes z}{|z|^2} \right),$$

which implies, thanks to the convergence of $\Gamma_n(|z|)$, that $\sqrt{a_n(z)} \to \sqrt{a(z)}$ in $L_{loc}^2(\mathbb{R}^N)$ up to possibly passing to a subsequence in n. Since

$$\sqrt{f_n^n(v_*)(1 - \varepsilon_n f_n^n(v_*))} \to \sqrt{f_*} \text{ in } L_{loc}^2((0,\infty) \times \mathbb{R}_{xvv_*}^{3N})$$

and $\gamma_\delta(f_n^n) \to \gamma_\delta(f)$ in $L_{loc}^1((0,\infty) \times \mathbb{R}_{xvv_*}^{3N})$, we can deduce that

$$\sqrt{a_n(v - v_*)} \sqrt{f_n^n(v_*)(1 - \varepsilon_n f_n^n(v_*))} \gamma_\delta(f_n^n) \to \sqrt{a(v - v_*)} \sqrt{f_*} \gamma_\delta(f)$$

in $L_{loc}^1((0,\infty) \times \mathbb{R}_{xvv_*}^{3N})$.

Next, notice that the explicit formula (39) allows us to calculate the divergence of the matrix $\sqrt{a}$, giving

$$\text{div}_z \left[\sqrt{a_n(z)} \right] = -(N-1)\sqrt{\Gamma_n(|z|)} \frac{z}{|z|^2},$$

which converges to $\mathrm{div}_z(\sqrt{a(z)})$ in $L^1_{loc}((0,\infty) \times \mathbb{R}^{3N}_{xvv_*})$ and therefore we have that

$$(40) \quad \sqrt{a_n(v-v_*)}\sqrt{f^n_n(v_*)(1-\varepsilon_n f^n_n(v_*))}\nabla_v\gamma_\delta(f^n_n)$$
$$\rightarrow \mathrm{div}_v\left[\sqrt{a(v-v_*)}\sqrt{f_*}\gamma_\delta(f)\right] - \mathrm{div}_v\left[\sqrt{a(v-v_*)}\right]\sqrt{f_*}\gamma_\delta(f)$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{2N}_{x,v})$, and we notate the limit distribution as $\sqrt{a(v-v_*)}\sqrt{f_*}\nabla_v\gamma_\delta(f)$. Lemma 4 gives us that (36) is bounded in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$, which then implies that (38) is uniformly bounded in $L^2_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v}; L^2(\mathbb{R}^N_{v_*}))$. Hence the convergence (40) actually holds weakly in this space and the limit distribution $\sqrt{a(v-v_*)}\sqrt{f_*}\nabla_v\gamma_\delta(f)$ can be represented by a function in $L^2_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v}; L^2(\mathbb{R}^N_{v_*}))$.

This implies there exists some positive defect measure μ such that

$$\left|\sqrt{a_n(v-v_*)}\sqrt{f^n_n(v_*)(1-\varepsilon_n f^n_n(v_*))}\nabla_v\gamma_\delta(f^n_n)\right|^2 \rightharpoonup \left|\sqrt{a(v-v_*)}\sqrt{f_*}\nabla_v\gamma_\delta(f)\right|^2 + \mu$$

weakly in $L^1_{loc}((0,\infty) \times \mathbb{R}^{3N}_{xvv_*})$ and thus the quadratic term can be written as

$$\int_{v_*}\left|\sqrt{a(v-v_*)}\sqrt{f_*}\nabla_v\gamma_\delta(f)\right|^2.$$

6.2. Conservation laws and entropy inequality.

So far we have used the approximation schemes $(f^n_n)_n$ to show that f satisfies the renormalized Landau equation with defect measure. In this second part of the proof, we will use the fact that the $(f^n)_n$ are weak solutions of the LFD equation to conclude that f satisfies the conservation laws and inequalities of the Definition 1.

Notice that inequality (33) applied to the diagonal subsequence (35) implies in particular that

$$(41) \qquad\qquad f^n \rightarrow f \text{ in } L^1_{loc}((0,\infty), L^1(\mathbb{R}^{2N}_{x,v}))$$

and passing to a subsequence in n we conclude that $f^n(t) \rightarrow f(t)$ in $L^1_{loc}(\mathbb{R}^{2N}_{x,v})$ for almost every $t \geq 0$, hence we have

$$(42) \qquad\qquad \int_{xv} f(t) = \lim_{n\to\infty}\int_{xv} f^n(t) = \lim_{n\to\infty}\int_{xv} f^n_0 = \int_{xv} f_0,$$

which implies the conservation of mass. Next we have, from Fatou's lemma,

$$\int_{xv} f(t)\psi \leq \liminf_{n\to\infty}\int_{xv} f^n(t)\psi \leq \liminf_{n\to\infty}\int_{xv} f^n_0\psi = \int_{xv} f_0\psi,$$

for $\psi = |v|^2$ or $\psi = |x-tv|^2$, which implies the inequalities of kinetic energy and momentum of inertia, respectively. Finally, by interpolation, we find that

$$\int_{xv} f(t)v_i = \int_{xv} f_0 v_i,$$

which corresponds to the conservation of momentum.

6.3. Entropy inequality.

Finally, let us show that the entropy inequality holds for the limit solution f. Since f^n is a weak solution of LFD with quantum parameter ε_n and initial data f^n_0, it obeys the quantum entropy inequality,

$$(43) \quad \frac{1}{\varepsilon_n}\int_{xv} \varepsilon_n f^n \log(\varepsilon_n f^n) + (1-\varepsilon_n f^n)\log(1-\varepsilon_n f^n) + \int_0^t\int_{xv} d^n \leq$$
$$\frac{1}{\varepsilon_n}\int_{xv} \varepsilon_n f^n_0 \log(\varepsilon_n f^n_0) + (1-\varepsilon_n f^n_0)\log(1-\varepsilon_n f^n_0),$$

where d^n is the entropy dissipation (15) with quantum parameter ε_n, using the interpretation from Remark 4. From the conservation of mass (42) we can rewrite inequality (43) as

$$\int_{xv} f^n \log f^n + \frac{1 - \varepsilon_n f^n}{\varepsilon_n} \log(1 - \varepsilon_n f^n) + \int_0^t \int_{xv} d^n \le \int_{xv} f_0^n \log f_0^n + \frac{1 - \varepsilon_n f_0^n}{\varepsilon_n} \log(1 - \varepsilon_n f_0^n).$$

But since for every $0 \le x \le \varepsilon^{-1}$ one has

$$-x \le \frac{1 - \varepsilon x}{\varepsilon} \log(1 - \varepsilon x) \le -x + \varepsilon x^2,$$

the above inequality becomes

$$(44) \qquad \int_{xv} f^n \log f^n + \int_0^t \int_{xv} d^n \le \int_{xv} f_0^n \log f_0^n + \varepsilon_n \int_{xv} (f_0^n)^2,$$

where we have used once again the conservation (42).

Since $\varepsilon_n (f_0^n)^2 \le f_0^n$ we have from (19) and dominated convergence that the right-hand side converges to

$$\int_{xv} f_0 \log f_0.$$

From (41), we may extract a further subsequence such that

$$(45) \qquad\qquad f^n \to f \text{ a.e. in } (0, \infty) \times \mathbb{R}^{2N}_{x,v}$$

which then implies, from Fatou's lemma, that

$$(46) \qquad\qquad \int_{xv} f \log f + \liminf_{n \to \infty} \int_0^t \int_{xv} d^n \le \int_{xv} f_0 \log f_0.$$

So all that remains is to pass the entropy dissipation d^n to the limit. From Remark 4 we recall that, since we don't suppose any regularity of f in v, the $\int_0^t \int_{xv} d^n$ should be instead be viewed as a notation for $\|D^n\|_{L^2}^2$ of the norm of D^n in $L^2((0,t) \times \mathbb{R}^{3N}_{xvv_*})$, where D^n is the function

$$D^n = \operatorname{div}_v \left(\sqrt{a(v - v_*)} \sqrt{f_*^n (1 - \varepsilon_n f_*^n)} \frac{2}{\sqrt{\varepsilon_n}} \arcsin \sqrt{\varepsilon_n f^n} \right)$$

$$- \operatorname{div}_v \left(\sqrt{a(v - v_*)} \right) \sqrt{f_*^n (1 - \varepsilon_n f_*^n)} \frac{2}{\sqrt{\varepsilon_n}} \arcsin \sqrt{\varepsilon_n f^n}$$

$$- \operatorname{div}_{v_*} \left(\sqrt{a(v - v_*)} \sqrt{f^n (1 - \varepsilon_n f^n)} \frac{2}{\sqrt{\varepsilon_n}} \arcsin \sqrt{\varepsilon_n f_*^n} \right)$$

$$+ \operatorname{div}_{v_*} \left(\sqrt{a(v - v_*)} \right) \sqrt{f^n (1 - \varepsilon_n f^n)} \frac{2}{\sqrt{\varepsilon_n}} \arcsin \sqrt{\varepsilon_n f_*^n}$$

From (45) we can deduce that

$$(47) \qquad \sqrt{a(v - v_*)} \sqrt{f_*^n (1 - \varepsilon_n f_*^n)} \frac{\arcsin \sqrt{\varepsilon_n f^n}}{\sqrt{\varepsilon_n f^n}} \sqrt{f^n} \to \sqrt{a(v - v_*)} \sqrt{f f_*}$$

for almost every $(t, x, v, v_*) \in (0, \infty) \times \mathbb{R}^{3N}_{xvv_*}$. We also have that

$$\left| \sqrt{a(v - v_*)} \sqrt{f_*^n (1 - \varepsilon_n f_*^n)} \frac{\arcsin \sqrt{\varepsilon_n f^n}}{\sqrt{\varepsilon_n f^n}} \sqrt{f^n} \right| \le \frac{\pi}{2} \left| \sqrt{a(v - v_*)} \sqrt{f_*^n} \sqrt{f^n} \right|.$$

The convergence (41) implies there exists a function g in $L^1_{loc}((0, \infty), L^1(\mathbb{R}^{2N}_{x,v}))$ such that, passing to a subsequence in n, we have $|f^n| \le g$ uniformly in n. This in turn implies

$$\left| \sqrt{a(v - v_*)} \sqrt{f_*^n} \sqrt{f^n} \right| \le \left| \sqrt{a(v - v_*)} \sqrt{g_*} \sqrt{g} \right|.$$

Let us show that this function is in $L^1_{loc}((0,\infty) \times \mathbb{R}^{3N}_{xvv_*})$. Indeed, from uniqueness of the square root of positive matrices,

$$(48) \qquad \sqrt{a(z)} = \sqrt{\Gamma(|z|)} \left(I - \frac{z \otimes z}{|z|^2} \right),$$

which in view of (10) and $\sqrt{x} \le 1 + x$ for every $x \ge 0$, implies there exist functions $\alpha \in L^1(\mathbb{R}^N)$ and $\beta \in L^\infty(\mathbb{R}^N)$ such that $\sqrt{a(z)} = \alpha(z) + \beta(z)$. Consider the compact sets $K_{tx} \subset (0,\infty) \times \mathbb{R}^N_x$, $K_v \subset \mathbb{R}^N_v$ and $K_{v_*} \subset \mathbb{R}^N_{v_*}$, then

$$\int_{K_{tx} \times K_v \times K_{v_*}} \left| \sqrt{a(v-v_*)} \right|_{ij} \sqrt{g_*} \sqrt{g} \le \left(\|\alpha\|_{L^1} + \|\beta\|_{L^\infty} |K_v|^{1/2} |K_{v_*}|^{1/2} \right) \times$$
$$\times \|g\|^{1/2}_{L^1(K_{tx} \times K_v)} \|g\|^{1/2}_{L^1(K_{tx} \times K_{v_*})},$$

hence, by dominated convergence, (47) holds in $L^1_{loc}((0,\infty) \times \mathbb{R}^{3N}_{xvv_*})$.

Next, the explicit expression (48) allows us to compute the divergence of this matrix, leading

$$\operatorname{div}_z \left[\sqrt{a(z)} \right] = -(N-1)\sqrt{\Gamma(|z|)} \frac{z}{|z|^2}.$$

The integrability (10) allows us to deduce that $\operatorname{div}_z \sqrt{a(z)}$ is also in $L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ and as before we deduce that

$$(49) \qquad \operatorname{div}_v \left(\sqrt{a(v-v_*)} \right) \sqrt{f^n_*(1-\varepsilon_n f^n_*)} \frac{\arcsin\sqrt{\varepsilon_n f^n}}{\sqrt{\varepsilon_n f^n}} \sqrt{f^n} \to \operatorname{div}_v \left(\sqrt{a(v-v_*)} \right) \sqrt{f f_*}$$

in $L^1_{loc}((0,\infty) \times \mathbb{R}^{3N}_{xvv_*})$.

The convergences (47) and (49) imply that

$$(50) \quad \operatorname{div}_v \left(\sqrt{a(v-v_*)} \sqrt{f^n_*(1-\varepsilon_n f^n_*)} \frac{2}{\sqrt{\varepsilon_n}} \arcsin\sqrt{\varepsilon_n f^n} \right)$$
$$- \operatorname{div}_v \left(\sqrt{a(v-v_*)} \right) \sqrt{f^n_*(1-\varepsilon_n f^n_*)} \frac{2}{\sqrt{\varepsilon_n}} \arcsin\sqrt{\varepsilon_n f^n}$$
$$\to \operatorname{div}_v \left(\sqrt{a(v-v_*)} \sqrt{f f_*} \right) - \operatorname{div}_v \left(\sqrt{a(v-v_*)} \right) \sqrt{f f_*}$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{3N}_{xvv_*})$. Exchanging v and v_* and summing it to (50), we conclude that D^n converges to

$$D \equiv 2\operatorname{div}_v \left(\sqrt{a(v-v_*)} \sqrt{f f_*} \right) - 2\operatorname{div}_v \left(\sqrt{a(v-v_*)} \right) \sqrt{f f_*}$$
$$- 2\operatorname{div}_{v_*} \left(\sqrt{a(v-v_*)} \sqrt{f f_*} \right) + 2\operatorname{div}_{v_*} \left(\sqrt{a(v-v_*)} \right) \sqrt{f f_*}$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^{3N}_{xvv_*})$.

On the other hand, we have from (44) that

$$\|D^n\|^2_{L^2} = \int_0^t \int_{xv} d^n \le \int_{xv} f^n_0 \log f^n_0 + \int_{xv} f^n_0 - \int_{xv} f^n \log f^n,$$

and thus D^n is uniformly bounded in $L^2((0,\infty) \times \mathbb{R}^{3N}_{xvv_*})$ and we have $D^n \rightharpoonup D$ weakly in this space. Then, from the lower semi-continuity of the norm with respect to the weak convergence we have

$$\int_0^t \int_{xv} d \equiv \|D\|^2_{L^2} \le \liminf_{n\to\infty} \|D^n\|^2_{L^2} = \liminf_{n\to\infty} \int_0^t \int_{xv} d^n$$

which, together with (46), leads the result.

7. Proof of Theorem 2

It remains just to prove the convergence of the quadratic term $-\beta''_\delta(f_n^n)\overline{a}_{ij}^{n,n}\frac{\partial f_n^n}{\partial v_i}\frac{\partial f_n^n}{\partial v_j}$. As before, we can't prove this converges to something expressible as a function of the objects we have thus far and as a consequence this term which will be responsible for the appearance of a defect measure. Indeed let

$$\gamma_\delta(t) = -\frac{2\sqrt{2}}{\sqrt{\delta(1+\delta t)}},$$

and we rewrite, in the sense of distributions,

$$-\beta''_\delta(f_n^n)\overline{a}_{ij}^{n,n}\frac{\partial f_n^n}{\partial v_i}\frac{\partial f_n^n}{\partial v_j} = \overline{a}_{ij}^{n,n}\frac{\partial\gamma_\delta(f_n^n)}{\partial v_i}\frac{\partial\gamma_\delta(f_n^n)}{\partial v_j}$$

$$= \langle\overline{a}^{n,n}\nabla_v(\gamma_\delta(f_n^n)), \nabla_v(\gamma_\delta(f_n^n))\rangle$$

$$= \left|\sqrt{\overline{a}^{n,n}}\nabla_v(\gamma_\delta(f_n^n))\right|^2$$

Rewriting the function inside the absolute value as

$$(51) \qquad \left(\sqrt{\overline{a}^{n,n}}\right)_{ij}\frac{\partial\gamma_\delta(f_n^n)}{\partial v_j} = \frac{\partial}{\partial v_j}\left[\left(\sqrt{\overline{a}^{n,n}}\right)_{ij}\gamma_\delta(f_n^n)\right] - \frac{\partial\left(\sqrt{\overline{a}^{n,n}}\right)_{ij}}{\partial v_j}\gamma_\delta(f_n^n),$$

as before, we want to apply some weak-strong convergence result here to at least deduce the convergence of this term in the sense of distributions. However, it soon becomes clear that these are not directly applicable in this case, since the convergence properties of the matrix $\sqrt{\overline{a}^{n,n}}$ cannot be deduced directly from the convergence of $\overline{a}^{n,n}$ alone. For example, the singularities of the derivatives of the square root of a matrix when close to zero prevent us from concluding the convergence of the second term in (51).

What we can do then is approximate this matrix by one that is more regular and therefore we can deduce this term in the limit. We then follow an approach close to Villani's in [13] and consider the approximation of the square root $S^k(M) = \Lambda^k(M)\chi^k(M)\sqrt{M}$, where χ^k is a $C^\infty(\mathbb{S}_+^n)$ function, defined on the space of positive semi-definite matrices $\mathbb{S}_+^n$ such that

$$\chi^k(M) = \begin{cases} 1, & \text{if } \lambda_{max}(M) \leq k \text{ and } \lambda_{min}(M) \geq \frac{1}{k} \\ 0, & \text{if } \lambda_{max}(M) \geq 2k \text{ or } \lambda_{min}(M) \leq \frac{1}{2k} \end{cases}$$

where $\lambda_{min}(M)$ and λ_{max} denote respectively the smallest and the largest eigenvalue of M. In other words, we truncate $\sqrt{M}$ at high values and at the singularity at zero. We can show that S^k is a C_c^∞ function such that $S^k(M) \to M$ for every positive semi-definite matrix M. Also, the sequence $(S^k(M))_k$ is increasing with respect to the matrix order and we have $S^k(M) \leq \sqrt{M}$.

Then,

$$\left(S^k(\overline{a}^{n,n})^2\right)_{ij}\frac{\partial\gamma_\delta(f_n^n)}{\partial v_i}\frac{\partial\gamma_\delta(f_n^n)}{\partial v_j} = \left|S^k(\overline{a}^{n,n})\nabla_v(\gamma_\delta(f_n^n))\right|^2$$

and for the term inside the absolute value we have the convergence

$$\left(S^k(\overline{a}^{n,n})\right)_{ij}\frac{\partial\gamma_\delta(f_n^n)}{\partial v_j} = \frac{\partial}{\partial v_j}\left[\left(S^k(\overline{a}^{n,n})\right)_{ij}\gamma_\delta(f_n^n)\right] - \frac{\partial\left(S^k(\overline{a}^{n,n})\right)_{ij}}{\partial v_j}\gamma_\delta(f_n^n)$$

$$(52) \qquad\qquad\qquad\qquad \Big\downarrow$$

$$\frac{\partial}{\partial v_j}\left[\left(S^k(\overline{a})\right)_{ij}\gamma_\delta(f)\right] - \frac{\partial\left(S^k(\overline{a})\right)_{ij}}{\partial v_j}\gamma_\delta(f) \equiv \left(S^k(\overline{a})\right)_{ij}\frac{\partial\gamma_\delta(f)}{\partial v_j}$$

in the sense of distributions.

Indeed, for each k, S^k is a Lipschitz function, the convergence of $\overline{a}^{n,n} \to \overline{a}$ in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ implies that $S^k(\overline{a}^{n,n}) \to S^k(\overline{a})$ in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$. Moreover, the derivatives of S^k are also Lipschitz, implying the convergence of $\frac{\partial S^k(\overline{a}^{n,n})}{\partial v_j} \to \frac{\partial S^k(\overline{a})}{\partial v_j}$ in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$.

Notice that the equality with $\left(S^k(\overline{a})\right)_{ij} \frac{\partial \gamma_\delta(f)}{\partial v_j}$ in (52) is a definition of a notation, rather than the proof of this limit, so it's important to notice that this product of functions is not necessarily a pointwise one. One should rather see the above expression as a black box. We will also notate $S^k(\overline{a})\nabla_v \gamma_\delta(f)$ the vector such that the i-th component is equal to $\left(S^k(\overline{a})\right)_{ij} \frac{\partial \gamma_\delta(f)}{\partial v_j}$.

Even though we have defined a notation for the limit distribution rather than calculated, we can nevertheless show that this distribution can be represented by an $L^2_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$ function. Indeed, since $S^k(M) \leq \sqrt{M}$, we have that

$$(53) \qquad \left(S^k(\overline{a}^{n,n})^2\right)_{ij} \frac{\partial \gamma_\delta(f_n^n)}{\partial v_i} \frac{\partial \gamma_\delta(f_n^n)}{\partial v_j} \leq \overline{a}^{n,n}_{ij} \frac{\partial \gamma_\delta(f_n^n)}{\partial v_i} \frac{\partial \gamma_\delta(f_n^n)}{\partial v_j}$$

and from Lemma 4, this last term is uniformly bounded in $L^1_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$, which then implies $\left(S^k(\overline{a}^{n,n})\nabla_v(\gamma_\delta(f_n^n))\right)_n$ is uniformly bounded in $L^2_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$, for every k. We have, therefore

$$(54) \qquad S^k(\overline{a}^{n,n})\nabla_v(\gamma_\delta(f_n^n)) \rightharpoonup S^k(\overline{a})\nabla_v(\gamma_\delta(f)) \text{ in } L^2_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$$

and the distributions $S^k(\overline{a})\nabla_v \gamma_\delta(f)$ can be represented by L^2_{loc} functions.

This implies in particular that $|S^k(\overline{a}^{n,n})\nabla_v(\gamma_\delta(f_n^n)) - S^k(\overline{a})\nabla_v \gamma_\delta(f)|$ is uniformly bounded in $L^2_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$. Let then μ^k be a measure such that, passing to a subsequence in n,

$$(55) \qquad \left|S^k(\overline{a}^{n,n})\nabla_v(\gamma_\delta(f_n^n)) - S^k(\overline{a})\nabla_v \gamma_\delta(f)\right|^2 \rightharpoonup \mu^k,$$

and this way we also have that, for a compact set K,

$$\begin{aligned}
|\mu^k|(K) &\leq \liminf_{n\to\infty} \left\| \left|S^k(\overline{a}^{n,n})\nabla_v(\gamma_\delta(f_n^n)) - S^k(\overline{a})\nabla_v \gamma_\delta(f)\right|^2 \right\|_{L^1(K)} \\
&\leq 4\liminf_{n\to\infty} \|S^k(\overline{a}^{n,n})\nabla_v(\gamma_\delta(f_n^n))\|^2_{L^2(K)} \\
&\leq 4\liminf_{n\to\infty} \|\overline{a}^{n,n}\nabla_v(\gamma_\delta(f_n^n))\nabla_v(\gamma_\delta(f_n^n))\|_{L^1(K)}
\end{aligned}$$

where the last inequality comes from (53). Hence, $(\mu^k)_k$ is bounded in $\mathcal{M}_{loc}((0,\infty) \times \mathbb{R}^{2N}_{x,v})$, and thus extracting a subsequence in k we have that $\mu^k \rightharpoonup \mu$.

Lower semi-continuity of the L^2 norm with respect to weak convergence implies, using (54), that for every $T, R > 0$ we have

$$\int_{(0,T)\times B_R \times B_R} |S^k(\overline{a})\nabla_v(\gamma_\delta(f))|^2 \leq \int_{(0,T)\times B_R \times B_R} |S^k(\overline{a}^{n,n})\nabla_v(\gamma_\delta(f_n^n))|^2$$

and from (53) the right-hand side is uniformly bounded by a constant. Extracting a subsequence in k we impose

$$S^k(\overline{a})\nabla_v(\gamma_\delta(f)) \rightharpoonup \sqrt{a}\nabla_v(\gamma_\delta(f)),$$

that is, as before, the distribution $\sqrt{a}\nabla_v(\gamma_\delta(f))$ is a notation for the limit of the sequence $S^k(\overline{a})\nabla_v(\gamma_\delta(f))$, which can be represented by an L^2_{loc} function.

Once again, notice that the above definitions are just notations representing distributions and should not be seen as a pointwise product of functions, if we have no additional hypotheses about the regularity of the functions in question. If we knew that this was a pointwise product, we could

easily show the convergence of $S^k(\overline{a})\nabla_v\gamma_\delta(f)$ to $\sqrt{a}\nabla_v\gamma_\delta(f)$. Indeed that's the case for f_n^n. Since χ^k is an increasing sequence converging a.e. to the constant function 1 we have by Beppo-Levi that

$$(56) \qquad |S^k(\overline{a}^{n,n})\nabla_v(\gamma_\delta(f_n^n))|^2 = \chi^k(\overline{a}^{n,n})^2|\sqrt{\overline{a}^{n,n}}\nabla_v(\gamma_\delta(f_n^n))|^2 \to |\sqrt{\overline{a}^{n,n}}\nabla_v(\gamma_\delta(f_n^n))|^2$$

in $L_{loc}^1((0,\infty) \times \mathbb{R}_{x,v}^{2N})$.

This argument doesn't work directly with $S^k(\overline{a})\nabla_v\gamma_\delta(f)$, because this is not necessarily the pointwise product of $S^k(\overline{a})$ with $\nabla_v\gamma_\delta(f)$. However, the following result shows us that this convergence occurs even if we don't have this property.

Now, this result, combined with (55), (56) and the fact that $\mu^k \to \mu$ implies that

$$\left|\sqrt{\overline{a}^{n,n}}\nabla_v(\gamma_\delta(f_n^n))\right|^2 \rightharpoonup \left|\sqrt{a}\nabla_v\gamma_\delta(f)\right|^2 + \mu.$$

and then it suffices to notice that

$$\left|\sqrt{\overline{a}^{n,n}}\nabla_v(\gamma_\delta(f_n^n))\right|^2 = \overline{a}^{n,n}\nabla_v(\gamma_\delta(f_n^n))\nabla_v(\gamma_\delta(f_n^n)) = -\beta_\delta''(f_n^n)\overline{a}_{ij}^{n,n}\frac{\partial f_n^n}{\partial v_i}\frac{\partial f_n^n}{\partial v_j}$$

and to define $-\beta_\delta''(f)\overline{a}_{ij}\frac{\partial f}{\partial v_i}\frac{\partial f}{\partial v_j}$ as the square of the $L_{loc}^2((0,\infty) \times \mathbb{R}_{x,v}^{2N})$ function $\sqrt{a}\nabla_v(\gamma_\delta(f))$.

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